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STRUCTURAL DYNAMICS, STABILITY, AND CONTROL OF HELICOPTERS

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DYNAMIC SYNTHESIS AND MODAL CHARACTERISTICS OF HELICOPTERS

by

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Abstract

This investigation is concerned with the dynamic synthesis of gyroscopic structures consisting of point-connected substructures. The object is to develop a mathematical model capable of an adequate simulation of the modal characteristics of a helicopter using a minimum number of degrees of freedom. The basic approach is to regard the helicopter structure as an assemblage of flexible substructures. This approach has the advantage that it permits the representation of each substructure by a given number of "modes". The variational equations for the perturbed motion about certain equilibrium solutions are derived. Assuming that some of the substructures are continuous, the variational equations are of the hybrid type so that it is necessary to discretize them, a task done quite well in the context of the component-mode synthesis. The discretized variational equations can be conveniently exhibited in matrix form, and a great deal of information about the system modal characteristics can be extracted from the coefficient matrices. The derivation of the variational equations requires a monumental amount of algebraic operations. To automate this task a symbolic manipulation program on a digital computer is developed.

Contents

1. Introduction

This investigation is concerned with the dynamic synthesis of helicopters. The object is to develop a mathematical model capable of an adequate simulation of the modal characteristics of a helicopter using a minimum number of degrees of freedom.

The method of approach represents an extension of the component-mode synthesis (Refs. 1 and 2) to gyroscopic systems. The basic idea is to regard the helicopter structure as an assemblage of flexible substructures. In the case of a helicopter, the various substructures can be identified as the airframe, the transmission shaft, the main rotor, and the tail rotor. The rotor can be simulated as a single substructure or as a collection of flexible blades. To ensure that the various substructures are acting as a single system, an orderly kinematical procedure for point-connected substructures is developed (Ref. 3).

The substructure synthesis approach has the advantage that it permits the representation of each substructure by a given number of "modes". This number can be varied according to the stiffness properties of the substructure. Clearly, a stiff substructure can be represented by a smaller number of degrees of freedom than a flexible one. This is not the only advantage, however. Indeed, the approach has the added versatility that it permits a given substructure to be represented by a discrete mathematical model or a continuous one. For example, it is reasonable to represent the fuselage by a discrete model and a rotor blade by a continuous model. Assuming that some of the substructures are continuous, the system equations of motion are hybrid in the sense that they comprise both ordinary and partial differential equations. Note that even in the absence of discrete substructures the system includes ordinary differential equations describing rotational motion.

The equations of motion can be conveniently obtained by the Lagrangian approach. The general equations are nonlinear and solutions of hybrid systems of nonlinear differential equations are beyond the state of the art. Fortunately, general solutions are not really necessary, nor are they likely to be very informative. Indeed, quite often one is content with solutions in the neighborhood of given special solutions, where the latter are often called equilibrium solutions. The motion in the neighborhood of the equilibrium solutions is referred to as the perturbed motion. When the perturbed motion is small, the linearized equations

obtained by ignoring the higher-order terms are known as the variational equations (Ref. 4). Two special solutions of considerable interest in the area of helicopter dynamics are associated with hover and forward flight.

The variational equations are still of the hybrid type. For a computational treatment of the equations, it is necessary to discretize them. This can be done quite well in the context of the component-mode synthesis. The procedure amounts to representing a continuous substructure by a finite series of space-dependent modes multiplied by time-dependent generalized coordinates. The meaning of these modes represents a study in itself and there appears to be some measure of freedom in their selection. The discretized variational equations of motion can be conveniently exhibited in matrix form, and a great deal of information about the system modal characteristics can be extracted from the coefficient matrices.

In the case in which one of the substructures rotates uniformly in the equilibrium state, such as in the case of the helicopter rotor, the coefficient matrices of the variational equations are generally periodic functions of time. However, when the rotor possesses both inertial and structural symmetry, which is possible if the rotor has at least three blades, a coordinate transformation can reduce the periodic matrices of the variational equations to constant ones. The procedure is sometimes referred to as the method of multiblade coordinates (Ref. 5). The resulting constant-coefficient equations represent a typical linear gyroscopic system.

The derivation of the variational equations of motion and their reduction to a constant-coefficient system requires a monumental amount of algebraic operations and differentiations. In addition, there is the question of producing an ordering scheme to derive equations of consistent order (Ref. 6). This points to the necessity of automating the various derivations, a task accomplished by means of a symbolic manipulation program on a digital computer.

In this report, the above aspects are first treated in a somewhat general way and then applied to a simplified mathematical model of a helicopter.

2. Kinematical Considerations

Let us consider a general structure consisting of a central substructure C and a given number of appended substructures (see Fig. 1). The appended substructures can be of many types. However, for the purpose of this paper we shall confine our discussion to a few specific types, namely, flexible substructures rotating relative to the central substructure (type A) and chains of substructures composed of a flexible substructure rotating relative to the central substructure to which another flexible substructure B is connected (type AB). The various types are illustrated in the illustrative example of Sec. 9. Clearly, rigid substructures are simply special cases of flexible substructures obtained by ignoring the flexibility. In addition, although we recognize that there may be more than one appendage of each type, we shall confine our discussion to a representative one of each type, with summation implied over the entire number of substructures of the same type. The kinematical relationships pertinent to such general structures are presented in detail in Ref. 3 and are only reviewed briefly here.

Let us consider the inertial system of axes XYZ with the origin at 0 and identify a system of axes $x_C y_C z_C$ with the origin at an arbitrary point C of the central substructure. Then, denoting the velocity vector of point C relative to point 0 in terms of components along XYZ by $\dot{\underset{\sim}{w}}_{OC}$, the angular velocity of the frame $x_C y_C z_C$ relative to XYZ in terms of components along $x_C y_C z_C$ by $\underset{\sim}{\Omega}_C$, and the matrix of direction cosines between $x_C y_C z_C$ and XYZ by T_{CO}, the absolute velocity of any mass point in the central substructure in terms of components along $x_C y_C z_C$ is

$$\dot{\underset{\sim}{w}}_C = T_{CO}\dot{\underset{\sim}{w}}_{OC} + \tilde{\Omega}_C(\underset{\sim}{r}_C + \underset{\sim}{u}_C) + \dot{\underset{\sim}{u}}_C = T_{CO}\dot{\underset{\sim}{w}}_{OC} - (\tilde{r}_C + \tilde{u}_C)\underset{\sim}{\Omega}_C + \dot{\underset{\sim}{u}}_C \quad (1)$$

In Eq. (1) $\underset{\sim}{r}_C$ is the position vector and $\underset{\sim}{u}_C$ is the elastic displacement vector of the mass point measured relative to $x_C y_C z_C$, $\tilde{\Omega}_C$ is a skew symmetric matrix associated with $\underset{\sim}{\Omega}_C$ (see Ref. 4), $\tilde{r}_C + \tilde{u}_C$ is a skew symmetric matrix associated with $\underset{\sim}{r}_C + \underset{\sim}{u}_C$, and $\dot{\underset{\sim}{u}}_C$ is the elastic velocity of the point relative to axes $x_C y_C z_C$.

Next, we consider the appended substructure A attached to the central substructure at point A and rotating relative to it. First, we introduce a nonrotating set of axes $x_{CA} y_{CA} z_{CA}$ in substructure C with origin at the point A and moving relative to axes $x_C y_C z_C$ as substructure C deforms. Then, introducing another set of axes $x_A y_A z_A$ attached to

substructure A with origin at A and moving with substructure A relative to substructure C, we can write the absolute velocity of any mass point in substructure A in terms of its elastic velocity relative to the frame $x_A y_A z_A$, the motion of the frame $x_A y_A z_A$ relative to axes $x_{CA} y_{CA} z_{CA}$, and the absolute motion of axes $x_{CA} y_{CA} z_{CA}$. To this end, we recognize that the absolute velocity of the attachment point A is obtained simply by evaluating Eq. (1) at the specific point A. We denote this velocity by $\dot{\underset{\sim}{w}}_{CA}$. The absolute angular velocity of axes $x_{CA} y_{CA} z_{CA}$ is

$$\underset{\sim}{\Omega}_{CA} = L_{EA} L_{GA} \underset{\sim}{\Omega}_C + \dot{\underset{\sim}{\beta}}_{CA} \tag{2}$$

where L_{GA} is the matrix of direction cosines between axes $x_{CA} y_{CA} z_{CA}$ and $x_C y_C z_C$ due to geometry alone, L_{EA} is the matrix of direction cosines between these axes due to the deformation of substructure C, and $\dot{\underset{\sim}{\beta}}_{CA}$ is the angular velocity of $x_{CA} y_{CA} z_{CA}$ relative to $x_C y_C z_C$ due to the deformation of substructure C. Explicitly, $\dot{\underset{\sim}{\beta}}_{CA}$ is the time derivative of the linear rotation vector

$$\underset{\sim}{\beta}_{CA} = L_{GA} (\tilde{\nabla}_C \underset{\sim}{u}_{CA}) \tag{3}$$

of axes $x_{CA} y_{CA} z_{CA}$ relative to axes $x_C y_C z_C$ in which $\tilde{\nabla}_C$ is the skew symmetric operator matrix corresponding to the curl operator associated with the rectangular coordinate system $x_C y_C z_C$; the subscript A of $\underset{\sim}{u}_{CA}$ specifies that the quantity in parenthesis is to be evaluated at point A. In addition, the matrix L_{EA} has the form

$$L_{EA} = 1 + \tilde{\beta}^T_{CA} \tag{4}$$

where 1 is the unit matrix and $\tilde{\beta}_{CA}$ is the skew symmetric matrix associated with the vector $\underset{\sim}{\beta}_{CA}$. Denoting by $\underset{\sim}{\omega}_A$ the angular velocity of body axes $x_A y_A z_A$ relative to axes $x_{CA} y_{CA} z_{CA}$ and letting L_A be the matrix of direction cosines between these axes, the absolute angular velocity of axes $x_A y_A z_A$ in terms of components along these axes can be written in the form

$$\underset{\sim}{\Omega}_A = L_A \underset{\sim}{\Omega}_{CA} + \underset{\sim}{\omega}_A = T_{AC} \underset{\sim}{\Omega}_C + L_A \dot{\underset{\sim}{\beta}}_{CA} + \underset{\sim}{\omega}_A \tag{5}$$

where

$$T_{AC} = L_A L_{EA} L_{GA} \tag{6}$$

is the matrix of direction cosines between axes $x_A y_A z_A$ and $x_C y_C z_C$. Finally, by analogy with Eq. (1) the absolute velocity of an arbitrary point in A is

$$\dot{\underset{\sim}{w}}_A = T_{AC} \dot{\underset{\sim}{w}}_{CA} + \tilde{\Omega}_A (\underset{\sim}{r}_A + \underset{\sim}{u}_A) + \dot{\underset{\sim}{u}}_A = T_{AC} \dot{\underset{\sim}{w}}_{CA} - (\tilde{r}_A + \tilde{u}_A) \underset{\sim}{\Omega}_A + \dot{\underset{\sim}{u}}_A \tag{7}$$

Turning our attention to the specific chain of substructures, we recognize that the first substructure in the chain is exactly analogous to a type A substructure. To describe the motion of the second substructure, we first consider the reference frame $x_{AB}y_{AB}z_{AB}$ in substructure A with origin at the point of connection, point B, between the second substructure B and the first substructure A and which moves relative to the frame $x_Ay_Az_A$ as substructure A deforms (see Fig. 1). Next, introducing the axes $x_By_Bz_B$ with the origin at point B and attached to substructure B, we can write the absolute velocity of any mass point in substructure B in terms of its velocity relative to the frame $x_By_Bz_B$, the motion of the frame $x_By_Bz_B$ relative to $x_{AB}y_{AB}z_{AB}$, and the absolute motion of the frame $x_{AB}y_{AB}z_{AB}$. By analogy with Eq. (5), we can write the absolute angular velocity of axes $x_By_Bz_B$ in terms of components along these axes in the form

$$\underset{\sim}{\Omega}_B = T_{BA}\underset{\sim}{\Omega}_A + L_B\dot{\underset{\sim}{\beta}}_{AB} + \underset{\sim}{\omega}_B \tag{8}$$

where $\underset{\sim}{\Omega}_A$ is given by Eq. (5), $\underset{\sim}{\omega}_B$ is the angular velocity of axes $x_By_Bz_B$ relative to $x_{AB}y_{AB}z_{AB}$, $T_{BA} = L_BL_{EB}L_{GB}$ is the matrix of direction cosines between axes $x_By_Bz_B$ and $x_Ay_Az_A$ in which L_{GB} is due to the geometry and L_{EB} is due to the deformation of body A, and by analogy with Eq. (3).

$$\underset{\sim}{\beta}_{AB} = L_{GB}(\tilde{\nabla}_B\underset{\sim}{u}_{AB}) \tag{9}$$

is the linear rotation vector of axes $x_{AB}y_{AB}z_{AB}$ relative to $x_Ay_Az_A$. Moreover, by analogy with Eq. (7), the absolute velocity of an arbitrary point in B is

$$\dot{\underset{\sim}{w}}_B = T_{BA}\dot{\underset{\sim}{w}}_{AB} + \tilde{\Omega}_B(\underset{\sim}{r}_B + \underset{\sim}{u}_B) + \dot{\underset{\sim}{u}}_B = T_{BA}\dot{\underset{\sim}{w}}_{AB} - (\tilde{r}_B + \tilde{u}_B)\underset{\sim}{\Omega}_B + \dot{\underset{\sim}{u}}_B \tag{10}$$

where $\dot{\underset{\sim}{w}}_{AB}$ is obtained by evaluating Eq. (7) at the point B and $\dot{\underset{\sim}{u}}_B$ is the elastic velocity relative to axes $x_By_Bz_B$.

3. System Lagrange's Equations in General Hybrid Form

The object is to derive the system Lagrange's equations of motion in general form. To this end, it is necessary to determine the functional dependence of the Lagrangian. If summation over all substructures is implied, then we can write the system kinetic energy in the general form

$$T = \frac{1}{2}\int_{m_C} \dot{\underset{\sim}{w}}_C^T\dot{\underset{\sim}{w}}_C dm_C + \frac{1}{2}\int_{m_A} \dot{\underset{\sim}{w}}_A^T\dot{\underset{\sim}{w}}_A dm_A + \frac{1}{2}\int_{m_B} \dot{\underset{\sim}{w}}_B^T\dot{\underset{\sim}{w}}_B dm_B \tag{11}$$

where $\dot{\underset{\sim}{w}}_C$, $\dot{\underset{\sim}{w}}_A$, and $\dot{\underset{\sim}{w}}_B$ are given by Eqs. (1), (7), and (10), respectively. Assuming that in equilibrium substructure A and substructure B rotate with the uniform angular velocities ω_A about z_A and ω_B about z_B, respectively, while any other motion is zero, we can write

$$
\begin{aligned}
\underset{\sim}{\Omega}_C &= \theta_C \dot{\underset{\sim}{\theta}}_C \\
\underset{\sim}{\omega}_A &= \omega_A \underset{\sim}{\ell}_A + \theta_A \dot{\underset{\sim}{\theta}}_A \qquad (12) \\
\underset{\sim}{\omega}_B &= \omega_B \underset{\sim}{\ell}_B + \theta_B \dot{\underset{\sim}{\theta}}_b
\end{aligned}
$$

where $\underset{\sim}{\ell}_A$ is the vector of direction cosines between z_A in the equilibrium configuration and axes $x_A y_A z_A$ and $\underset{\sim}{\ell}_B$ is the vector of direction cosines between z_B in the equilibrium configuration and axes $x_B y_B z_B$. Moreover, θ_C, θ_A, and θ_B are 3×3 matrices depending on the oscillation of axes $x_C y_C z_C$ relative to XYZ, $x_A y_A z_A$ relative to $x_{CA} y_{CA} z_{CA}$, and $x_B y_B z_B$ relative to $x_{AB} y_{AB} z_{AB}$, respectively, and $\dot{\underset{\sim}{\theta}}_C$, $\dot{\underset{\sim}{\theta}}_A$, and $\dot{\underset{\sim}{\theta}}_B$ are vectors of generalized velocities. Hence, the functional dependence of the kinetic energy is

$$
\begin{aligned}
T = T(&\dot{\underset{\sim}{w}}_{OC}, \theta_{Ci}, \dot{\underset{\sim}{\theta}}_C, \underset{\sim}{u}_C, \dot{\underset{\sim}{u}}_C, \underset{\sim}{u}_{CA}, \dot{\underset{\sim}{u}}_{CA}, \theta_{Ai}, \dot{\underset{\sim}{\theta}}_A, \underset{\sim}{u}_A, \dot{\underset{\sim}{u}}_A, \underset{\sim}{u}_{AB}, \dot{\underset{\sim}{u}}_{AB}, \\
&\theta_{Bi}, \dot{\underset{\sim}{\theta}}_B, \underset{\sim}{u}_B, \dot{\underset{\sim}{u}}_B, t) \qquad (13)
\end{aligned}
$$

where $\underset{\sim}{u}_{CA}$ denotes $\underset{\sim}{u}_C$ evaluated at point A of substructure C and $\underset{\sim}{u}_{AB}$ denotes $\underset{\sim}{u}_A$ evaluated at point B of substructure A.

Assuming that the potential energy is due entirely to elastic effects (including possible elastic restraints between substructures), we shall write it in the general form

$$
\begin{aligned}
V = U(\theta_{A1}, \theta_{A2}, \theta_{B1}, \theta_{B2}) &+ \frac{1}{2}\int_{D_C} \underset{\sim}{u}_C^T A_{DC} \underset{\sim}{u}_C dD_C + \frac{1}{2}\int_{S_C} \underset{\sim}{u}_C^T A_{SC} \underset{\sim}{u}_C dS_C \\
&+ \frac{1}{2}\int_{D_A} \underset{\sim}{u}_A^T A_{DA} \underset{\sim}{u}_A dD_A + \frac{1}{2}\int_{S_A} \underset{\sim}{u}_A^T A_{SA} \underset{\sim}{u}_A dS_A + \frac{1}{2}\int_{D_B} \underset{\sim}{u}_B^T A_{DB} \underset{\sim}{u}_B dD_B \\
&+ \frac{1}{2}\int_{S_B} \underset{\sim}{u}_B^T A_{SB} \underset{\sim}{u}_B dS_B \qquad (14)
\end{aligned}
$$

where U represents the potential energy due to elastic restraints of the rotation of substructure A relative to C and substructure B relative to A and A_{DC}, A_{SC}, A_{DA}, A_{SA}, A_{DB} and A_{SB} are "two-sided" differential operators, containing partial derivatives with respect to the spatial variables. The operators are symmetric and positive definite and can be identified as <u>energy operators</u> over the respective domains. The last six terms of Eq. (14) represent symbolic operations involving integrations

over the elastic domains D_C, D_A, and D_B and their boundaries S_C, S_A, and S_B, respectively.

Next, introduce the Lagrangian L = T-V and write it in the form

$$L = \int_{D_C} \hat{L}_{DC} dD_C + \int_{D_A} \hat{L}_{DA} dD_A + \int_{D_B} \hat{L}_{DB} dD_B + \int_{S_C} \hat{L}_{SC} dS_C + \int_{S_A} \hat{L}_{SA} dS_A + \int_{S_B} \hat{L}_{SB} dS_B - U \tag{15}$$

where $\hat{L}_{DC}$, $\hat{L}_{SC}$, $\hat{L}_{DA}$, $\hat{L}_{SA}$, $\hat{L}_{DB}$, and $\hat{L}_{SB}$ are Lagrangian densities. Using Hamilton's principle (Ref. 4) and following the standard procedure, we obtain Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\underset{\sim}{w}}_{OC}}\right) = \underset{\sim}{0} \quad , \tag{16a}$$

$$\frac{\partial L}{\partial \theta_{Ci}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_{Ci}}\right) = 0 \ , \quad \frac{\partial L}{\partial \theta_{Ai}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_{Ai}}\right) = 0 \ ,$$

$$\frac{\partial L}{\partial \theta_{Bi}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_{Bi}}\right) = 0 \qquad i=1,2,3 \tag{16b}$$

$$\frac{\partial \hat{L}_{DC}}{\partial \underset{\sim}{u}_C} - \frac{\partial}{\partial t}\left(\frac{\partial \hat{L}_{DC}}{\partial \dot{\underset{\sim}{u}}_C}\right) + \underset{\sim}{\mathcal{L}}_C \underset{\sim}{u}_C = \underset{\sim}{0} \text{ over } D_C$$

$$\frac{\partial \hat{L}_{DA}}{\partial \underset{\sim}{u}_A} - \frac{\partial}{\partial t}\left(\frac{\partial \hat{L}_{DA}}{\partial \dot{\underset{\sim}{u}}_A}\right) + \underset{\sim}{\mathcal{L}}_A \underset{\sim}{u}_A = \underset{\sim}{0} \text{ over } D_A \tag{16c}$$

$$\frac{\partial \hat{L}_{DB}}{\partial \underset{\sim}{u}_B} - \frac{\partial}{\partial t}\left(\frac{\partial \hat{L}_{DB}}{\partial \dot{\underset{\sim}{u}}_B}\right) + \underset{\sim}{\mathcal{L}}_B \underset{\sim}{u}_B = \underset{\sim}{0} \text{ over } D_B$$

where $\underset{\sim}{\mathcal{L}}_C$, $\underset{\sim}{\mathcal{L}}_A$, and $\underset{\sim}{\mathcal{L}}_B$ are linear differential operator vectors reflecting stiffness properties. Equations (16c) are subject to appropriate boundary conditions, which are omitted for brevity. Note that the relation between the operators A and $\tilde{\mathcal{L}}$ is given by

$$\int_D \underset{\sim}{u}^T \underset{\sim}{A}_D u dD + \int_S \underset{\sim}{u}^T \underset{\sim}{A}_S u dS = \int_D \underset{\sim}{u}^T \underset{\sim}{\mathcal{L}} \underset{\sim}{u} dD \tag{17}$$

where the satisfaction of the boundary conditions is implied.

4. Variational Equations

The hybrid set of equations, Eqs. (16), is generally nonlinear, particularly in the rotational coordinates. Moreover, due to the uniform angular velocity ω_A, the angle $\omega_A t$ appears in the matrix of direction cosines giving the orientation of axes $x_A y_A z_A$, and a similar statement can be made for axes $x_B y_B z_B$. Therefore, Eqs. (16) possess time-dependent coefficients, or more specifically periodic coefficients. General solutions of sets of hybrid nonlinear differential equations with periodic coefficients do not exist. However, our interest is in solutions in the neighborhood of known solutions. These known solutions are either constant solutions or periodic solutions and define an equilibrium state or equilibrium motion, respectively. To study the dynamic characteristics in the neighborhood of known solutions, the perturbation equations are derived. The linearized perturbation equations are known as variational equations. To derive the variational equations, one can expand the Lagrangian in a Taylor series. The linear terms define the equilibrium and can be ignored since they cancel out in the equations of motion. The quadratic terms lead to the linear terms in the equations of motion and are the only terms retained in the Lagrangian. Note that this procedure involves simply a transformation of coordinates so that the equilibrium of the new equations is trivial. However, the Lagrangian obtained in this way will still possess periodic coefficients. This creates a host of problems, as far as the analysis of the variational equations is concerned. Indeed, systems of equations with periodic coefficients are considerably more difficult to analyze and the results considerably harder to interpret compared with systems with constant coefficients. Under certain circumstances, it is possible to reduce a system with periodic coefficients in time to one with constant coefficients. Such systems are known as "reducible" and the reduction process involves a linear transformation, where the equations of the transformation also have periodic coefficients with the same period as the time-dependent terms to be eliminated. However, there is no general analytical procedure for determining the linear transformation leading to the constant-coefficients system. We shall not be concerned with a general reducible system but with a special kind, namely, one for which a real transformation exists such that the transformed Lagrangian remains real

and does not depend explicitly on time. Transformations with these properties can easily be found if the rotating substructures A and B possess certain symmetry. For example, if substructure B consists of three or more identical beams with one end fixed to a hub and spaced equally around the hub in a plane perpendicular to the axis of rotation, and substructure A consists of a uniform shaft rigidly attached to the hub, then the transformation can be found by the method of multiblade coordinates (Ref. 5). In the following, we shall assume that transformations $\underset{\sim}{u}^*_A = \underset{\sim}{u}^*_A(\underset{\sim}{u}_A, t)$, $\underset{\sim}{u}^*_B = \underset{\sim}{u}^*_B(\underset{\sim}{u}_B, t)$ exist such that the transformed Lagrangian does not depend explicitly on time. Taking into account the above discussion, the linear constant-coefficients equations of motion can be written in a form resembling Eqs. (16). In fact, the form remains the same if the generalized coordinates entering into Eqs. (16) can be regarded as measuring displacements from equilibrium.

Next, we must recognize that a system of the type under consideration possesses ignorable coordinates. Our object being to study the modal characteristics of the system, it is desirable to eliminate these ignorable coordinates. To this end, the three components of translation are recognized immediately as ignorable coordinates. It is not immediately apparent, however, that certain rotational coordinates may also be ignorable. Indeed, if we assume that in equilibrium the z_C axis is aligned with the z_A axis, then the angles θ_{C3} and θ_{A3}, defined as the rotation of substructure C about the z_C axis and the deviation of the rotation of substructure A about the z_A axis from steady spin, respectively, are recognized immediately as being ignorable coordinates. Since the generalized momenta associated with ignorable coordinates are conserved, we can regard the conservation of these generalized momenta as constraints on the system and use these constraints to eliminate the vector $\underset{\sim}{w}_{OC}$ and the rotational coordinates θ_{C3} and θ_{A3} from the formulation. A similar discussion leads to the elimination of θ_{B3}. In view of these constraints, the final variational equations can be written in a form resembling Eqs. (16b) and (16c) if in Eqs. (16b) we delete i=3.

5. System Discretization

The system of equations (16b) and (16c) is hybrid and not very convenient for studying its dynamic characteristics. The reason can be

easily traced to the fact that some of the coordinates depend not only on time but also on the spatial position. Hence, we shall find it necessary to eliminate the spatial dependence. This we shall do by a process referred to as series discretization. Before proceeding with the discretization, we shall present certain mathematical preliminaries.

Let us consider a function u that is continuous in the closed domain $\bar{D}=D+S$, where D denotes the open domain and S its boundary. The function is assumed to be quadratically integrable in the Lebesgue sense, which implies that the integral

$$||u||^2 = (u, u) = \int_D u^2 dD \tag{18}$$

exists, where $||u||$ is known as the norm of u. Our interest lies in approximating the function u by some other function. To this end, we consider the sequence of functions $u_1, u_2, \ldots, u_N, \ldots$ defined over D. Then, if $\lim_{N\to\infty} ||u_N - u|| = 0$, it is said that u_N converges in the mean to u.

Next, let us assume that the function u can be represented in the domain D by the series $u = \sum_{j=1}^{\infty} a_j\phi_j$, which is convergent in the mean, and consider the partial sum

$$u_N = \sum_{j=1}^{N} a_j\phi_j \tag{19}$$

Then, the set of functions ϕ_j is said to be complete if it is possible to find an integer N and a set of coefficients $a_1, a_2, \ldots, a_N$ such that for any $\varepsilon>0$, $||u - u_N|| < \varepsilon$. The question remains as to the nature of the functions ϕ_j $(j=1,2,\ldots)$. To answer this question, we wish to delve a little deeper into the properties of the differential operators A and L introduced earlier in conjunction with vector functions. In the first place, we should point out that, whereas L is an actual differential operator, A is an operator only in a symbolic sense. If L is a self-adjoint operator, then for any two functions in the field of definition of L

$$(Lu,v) = \int_D vLu dD = \int_D uLv dD = (u, Lv) \tag{20}$$

Integrating by parts, and considering the boundary conditions of the problem, we can write (see analogy with Eq. (17))

$$\int_D vLudD = \int_D vA_D udD + \int_S vA_S udS \qquad (21)$$

where A_D and A_S are "two-sided" symbolic operators symmetric in u and v. For convenience, we shall use the notation

$$[u, v]_A = \int_D vA_D udD + \int_S vA_S udS \qquad (22)$$

The operators L and A are said to be positive definite if the inequality $[u, u]_A > 0$ is satisfied.

In our particular case, $[u, u]_A$ defines twice the potential energy associated with a given closed domain $\bar{D}$. Then, the quantity $\| u \|_A$ defined by

$$\| u \|_A^2 = [u, u]_A \qquad (23)$$

is known as the <u>energy norm</u> of u. When it is clear that the norm involves the operator A, the subscript A can be ignored. The sequence of functions $u_1, u_2, \ldots, u_N, \ldots$ <u>converges in energy</u> to u if $\lim_{N\to\infty} \| u_N - u \| = 0$. If u_N represents the partial sum (19), then if for any $\varepsilon > 0$ it is possible to find an integer N and a set of coefficients $a_1, a_2, \ldots, a_N$ such that $\| u - u_N \| < \varepsilon$, the set of functions $\phi_1, \phi_2, \ldots, \phi_N, \ldots$ is said to be <u>complete in energy</u>.

To generalize these ideas, let L be a positive definite operator defined on some Hilbert space H with the scalar product (u, v) and the norm defined by Eq. (18). But the scalar product (22) also defines a Hilbert space. The Hilbert space defined by the scalar product (22) is called complete if any sequence ϕ_j of its elements satisfying the condition $\lim_{k,n\to\infty} \| \phi_k - \phi_n \| = 0$ has a limit (e.g. ϕ is the limit of the sequence ϕ_j if $\lim_{n\to\infty} \| \phi - \phi_n \| = 0$); otherwise it is called incomplete. In general, it is incomplete and must be made complete by adding certain new elements to it where the added elements are called limiting elements and the limiting element ϕ is defined by the sequence ϕ_j. The scalar product of the limiting elements ϕ and ψ is defined by

$$[\phi, \psi] = \lim_{n\to\infty} [\phi_n, \psi_n] \qquad (24)$$

and the norm of the limiting element ϕ by

$$\| \phi \| = \lim_{n\to\infty} \| \phi_n \| \qquad (25)$$

The new Hilbert space thus constructed is called the energy space and is denoted by H_A. Note that Eqs. (24) and (25) extend the definition of scalar product and energy norm to include functions which are not in the field of definition of the operator L. Moreover, because L is positive definite, all elements in the space H_A also belong to the initial Hilbert space H and convergence of a sequence in energy implies that it also converges in the mean. Hence, the functions ϕ_j in the partial sum (19) belong to the space H_A. The question is as to what distinguishes the functions belonging to H_A from those in the field of definition of L. It is clear that the integration by parts leading to Eq. (21) lowers the order of the operator involved by a factor of two. Hence, if L is of order 2p, then A_D is of order p, so that the requirements on the differentiability of the functions ϕ_j are lowered accordingly. More importantly, however, the energy integrals, Eq. (22), take automatically into account the natural boundary conditions, so that the functions ϕ_j need satisfy only the geometric boundary conditions. Such functions are sometimes referred to as energy functions. In more familiar terminology, we shall refer to functions belonging to H_A as admissible functions and those in the field of definition of L as comparison functions (Ref. 7).

The above concepts can be extended to a vector function $\underset{\sim}{u}$, by simply defining the norm as $\|\underset{\sim}{u}\|^2 = (\underset{\sim}{u}, \underset{\sim}{u}) = \int_D \underset{\sim}{u}^T\underset{\sim}{u}dD$ as well as the energy norm $|\!|\!|\underset{\sim}{u}|\!|\!|^2 = [\underset{\sim}{u}, \underset{\sim}{u}] = \int_D \underset{\sim}{u}^T\underset{\sim}{A}_D\underset{\sim}{u}dD + \int_S \underset{\sim}{u}^T\underset{\sim}{A}_S\underset{\sim}{u}dS$ where once again the integrals represent symbolic operations.

In view of the above, we shall represent the elastic motions in the domains D_C, D_A, and D_B by the series

$$\underset{\sim}{u}_C = \sum_{j=1}^{N_C} \underset{\sim}{\phi}_{Cj}\eta_{Cj} = \Phi_C\underset{\sim}{\eta}_C, \quad \underset{\sim}{u}_A = \sum_{j=1}^{N_A} \underset{\sim}{\phi}_{Aj}\eta_{Aj} = \Phi_A\underset{\sim}{\eta}_A,$$

$$\underset{\sim}{u}_B = \sum_{j=1}^{N_B} \underset{\sim}{\phi}_{Bj}\eta_{Bj} = \Phi_B\underset{\sim}{\eta}_B \tag{26}$$

where Φ_C, Φ_A, and Φ_B are $3\times N_C$, $3\times N_A$, and $3\times N_B$ matrices of space-dependent admissible functions and $\underset{\sim}{\eta}_C$, $\underset{\sim}{\eta}_A$, and $\underset{\sim}{\eta}_B$ are vectors of time-dependent generalized coordinates. This permits us to introduce the n-dimensional configuration vector $q = [\theta_{C1}\theta_{C2}\theta_{A1}\theta_{A2}\theta_{B1}\theta_{B2}\underset{\sim}{\eta}_C^T\underset{\sim}{\eta}_A^T\underset{\sim}{\eta}_B^T]^T$, where $n = 4+N_C+N_A+N_B$, which consists of generalized displacements measured from

equilibrium. Then, the system kinetic energy can be written as

$$T = \frac{1}{2}\dot{q}^T M \dot{q} + q^T F \dot{q} + \frac{1}{2} q^T K_T q + R^T \dot{q} + S^T q + \text{constants} \tag{27}$$

in which M and K_T are symmetric matrices. Recognizing that Eq. (27) represents a Taylor series expansion of the kinetic energy T about an equilibrium state, we conclude that the last three terms in Eq. (27) actually define the equilibrium state, with the inference that R is a constant vector and S is the n-dimensional zero vector. In addition, the constants in Eq. (27) can be ignored because they do not enter into the equations of motion. Hence, the last three terms in Eq. (27) can simply be omitted.

In terms of the system configuration vector q, the system potential energy takes the quadratic form

$$V = \frac{1}{2} q^T K_V q \tag{28}$$

where

$$K_V = \text{block-diag}\,\Big[0 \;\Big|\; \frac{\partial^2 U}{\partial\theta_{Ai}\,\partial\theta_{Aj}} \;\Big|\; \frac{\partial^2 U}{\partial\theta_{Bi}\partial\theta_{Bj}} \;\Big|\; \int_{D_C} \Phi_C^T A_{DC} \Phi_C dD_C + \int_{S_C} \Phi_C^T A_{SC} \Phi_C dS_C \;\Big|\; \int_{D_A} \Phi_A^T A_{DA} \Phi_A dD_A + \int_{S_A} \Phi_A^T A_{SA} \Phi_A dS_A + \;\Big|\; \int_{D_B} \Phi_B^T A_{DB} \Phi_B dD_B + \int_{S_B} \Phi_B^T A_{SB} \Phi_B dS_B\Big] \quad i,j = 1,\, 2 \tag{29}$$

is a symmetric matrix. Hence, the discretized equations of motion can be written in the matrix form

$$M\ddot{q} + (F^T - F)\dot{q} + (K_V - K_T)q = 0 \tag{30}$$

where M and K_T-K_V are positive definite symmetric matrices and F^T-F is a skew symmetric matrix. Equation (30) represents a typical gyroscopic system. The natural frequencies and natural modes of the complete structure and the closed-form solution of Eq. (30) can be obtained by methods such as those developed in Refs. 8 and 9. The interest here is not so much in the response as in the dynamic characteristics of the system, and in particular, the truncation effect on the predicted characteristics.

6. Inclusion Principle and Truncation Considerations

Following the procedure of Ref. 8, we introduce the 2n-dimensional state vector $x(t) = [\dot{q}^T(t)\ q^T(t)]^T$, as well as the 2n × 2n matrices

$$I = \begin{bmatrix} M & | & 0 \\ - & + & - \\ 0 & | & K_V - K_T \end{bmatrix}, \qquad G = \begin{bmatrix} F^T - F & | & K_V - K_T \\ - & + & - \\ K_T - K_V & | & 0 \end{bmatrix} \tag{31}$$

where 0 is the null matrix of order n and write the eigenvalue problem associated with the system (30) in the real symmetric form

$$\omega_r^2 I \underset{\sim}{y}_r = K \underset{\sim}{y}_r, \; \omega_r^2 I \underset{\sim}{z}_r = K \underset{\sim}{z}_r \qquad r = 1, 2, \ldots, n \tag{32}$$

where ω_r are the natural frequencies of oscillation of the system and $\underset{\sim}{y}_r$ and $\underset{\sim}{z}_r$ are the real eigenvectors, corresponding to the real and imaginary parts of $\underset{\sim}{x}_r$. Moreover, $K = G^T I^{-1} G$ is a real symmetric matrix. Assuming that I is positive definite, it follows that K is positive definite, so that the eigenvalues are not only real but positive. In addition, the eigenvalues ω_r^2 ($r=1, 2, \ldots, n$) have multiplicity two and because I and K are positive definite all the eigenvectors are independent and can be rendered mutually orthogonal with respect to the matrix I.

Next, let us use the Cholesky decomposition and write I in the form $I=LL^T$, where L is a lower triangular matrix. Introducing the notation $\underset{\sim}{y}_r' = L^T \underset{\sim}{y}_r$, $\underset{\sim}{z}_r' = L^T \underset{\sim}{z}_r$, ($r=1, 2, \ldots, n$), the eigenvalue problem (32) reduces to the standard form

$$\omega_r^2 \underset{\sim}{y}_r' = K' \underset{\sim}{y}_r', \; \omega_r^2 \underset{\sim}{z}_r' = K' \underset{\sim}{z}_r' \qquad r = 1, 2, \ldots, n \tag{33}$$

where $K'=L^{-1}KL^{-T}$ is a real symmetric positive definite matrix, in which $L^{-T}=(L^{-1})^T$.

Denoting by $\underset{\sim}{v}$ an arbitrary 2n-vector, Rayleigh's quotient associated with the eigenvalue problem (33) can be written in the form (Ref. 10)

$$R(\underset{\sim}{v}) = \underset{\sim}{v}^T K' \underset{\sim}{v} / \underset{\sim}{v}^T \underset{\sim}{v} \tag{34}$$

Because K' is real and symmetric, it is well known that Rayleigh's quotient has a stationary value in the neighborhood of an eigenvalue, so that a stationarity principle exists also for gyroscopic systems.

Now, we are in the position to discuss the truncation problem. The $2n \times 2n$ matrix K' was obtained as a result of representing the structure by an n-degree-of-freedom system. This representation is tantamount to the imposition of a given number of constraints on the original structure. For example, the constraints imposed on the system by the first of Eqs. (26) are $\eta_{C,N_C+1} = \eta_{C,N_C+2} = \ldots = 0$. Truncating the series for $\underset{\sim}{u}_C$ by

assuming that $\eta_{C,N_C} = 0$, we obtain a matrix K" obtained from K' by deleting two rows and the corresponding two columns. If the eigenvalues ω_r^2 of K' are such that $\omega_1^2 \leq \omega_2^2 \leq \ldots \leq \omega_n^2$ and the eigenvalues β_r^2 of K" are such that $\beta_1^2 \leq \beta_2^2 \leq \ldots \leq \beta_{n-1}^2$, then by the <u>inclusion principle</u> (Ref. 3), we have

$$\omega_1^2 \leq \beta_1^2 \leq \omega_2^2 \leq \beta_2^2 \leq \ldots \leq \omega_{n-1}^2 \leq \beta_{n-1}^2 \leq \omega_n^2 \tag{35}$$

Note that the fact that the eigenvalues of K' and K" have multiplicity two is automatically taken into account in inequalities (35). On the other hand, by relaxing one constraint, i.e. by adding one term to the series for $\underset{\sim}{u}_C$, we obtain a (2n+2)×(2n+2) matrix K'" which is obtained by adding two rows and columns to K'. The eigenvalues α_r^2 of K'" are such that

$$\alpha_1^2 \leq \omega_1^2 \leq \alpha_2^2 \leq \omega_2^2 \leq \ldots \leq \alpha_n^2 \leq \omega_n^2 \leq \alpha_{n+1}^2 \tag{36}$$

The above developments permit us to conclude that the <u>system natural frequencies tend to decrease monotonically with each additional degree of freedom</u>. At the same time there is a new frequency added which is higher than any of the previous ones. Inequalities (35) and (36) have been derived by substracting and adding one term to the series for $\underset{\sim}{u}_C$, i.e. the first of Eqs. (26). It is obvious that similar inequalities will be obtained by subtracting and adding one term to the series for $\underset{\sim}{u}_A$ or to the series for $\underset{\sim}{u}_B$. Hence, it is possible to assess the effects on the system natural frequencies of truncating each substructure individually.

It should be pointed out that the existence of an inclusion principle for gyroscopic systems is predicated on the existence of a stationarity principle for such systems.

7. <u>Selection of Admissible Functions</u>

The question remains as to how to select the admissible functions. The first thing that comes to mind is to select them as the eigenfunctions of the various substructures. This approach has two drawbacks: 1) the solution of the substructure eigenvalue problem may not be so easy to obtain and 2) an eigenvalue problem for the substructure may not be readily definable because of various rotational coordinates coupling the problem to that for a different substructure. Whereas substructure eigenfunctions (if they can be defined) would make a suitable set of admissible functions, fortunately, there are many other sets

of functions that can yield equally good results, provided the functions satisfy certain criteria, as discussed below.

The representation of substructure elastic motion in terms of admissible functions is a process closely related to the Rayleigh-Ritz method. Hence, it is natural to expect that the criteria for the selection of admissible functions for the Rayleigh-Ritz problem associated with a particular single structure should be equally valid here, where the structure is now a substructure of a larger assembled structure. Sufficient criteria for each closed domain $\bar{D}$ are (see Ref. 11):

1. Any finite number of admissible functions must be linearly independent.
2. The set of admissible functions must be complete in the energy space H_A of the domain $\bar{D}$.
3. The set of admissible functions should be strongly minimal in the energy space H_A of the domain $\bar{D}$.

The second of these criteria requires that we be able to identify a set of admissible functions that is complete in the energy space H_A. To this end, the following statement is useful: If A and B are positive definite operators and the spaces H_A and H_B contain the same elements, then any set that is complete in H_B is complete in H_A. Therefore, by letting B have a simple form for which a complete set is easily found, we can use that set of admissible functions for the operator A. This justifies the earlier statement that it is not necessary to use substructure eigenfunctions. Indeed, for an involved operator A it should be possible to find a very simple operator B such that H_A and H_B contain the same elements and a complete set of admissible functions for H_A is found readily without any of the complexities inherent in working with the operator A.

The third of the above criteria requires explanation. To this end, we consider an infinite system of admissible functions $\psi_1, \psi_2, \ldots$ in some space H. Then, the system $\psi_1, \psi_2, \ldots$ is said to be strongly minimal in H if the smallest eigenvalue of the matrix

$$R_n = [(\psi_i, \psi_j)_H]\Big|_{i,j=1}^{i,j=n} \tag{37}$$

is bounded below by a positive number that is independent of n. An orthonormal system is strongly minimal by definition because the eigenvalues of R_n are equal to one for all n. Moreover, if operators

A and B are self-adjoint and positive definite in some Hilbert space H and every member of H_A belongs to H_B, then any set of admissible functions in H_A which is strongly minimal in H_B is also strongly minimal in H_A. Hence, a system that is orthonormal in H_B is strongly minimal in H_A.

8. Discrete Substructures

In the preceding discussion, there was an implicit assumption that the substructures were continuous. The same approach, however, is valid for discrete substructures, as the dynamic characteristics are similar and do not depend on the mathematical model. Clearly, for discrete substructures no discretization is necessary but more often than not truncation is. For example, some discrete substructures can be represented by mathematical models possessing hundreds or even thousands of degrees of freedom. Yet it is not feasible to work with systems with such a large number of degrees of freedom and truncation is an absolute necessity. This can be done by representing the motion of the substructure by an adequate number of "modes". Typically, one would derive the eigenvalue problem for the substructure and attempt a solution, at least for a limited number of lower modes, and use these to represent the substructure. Of course, the question remains as to the "boundary conditions" to be imposed on the substructure. In the case of the central substructure one can regard it as entirely unrestrained. On the other hand, the choice of coordinates dictates that the substructure A be regarded as cantilevered at the origin A of axes $x_Ay_Az_A$.

If the substructure possesses a very large number of degrees of freedom, then the solution of the eigenvalue problem can become not only time consuming, but the results inaccurate. Then, the question arises whether it is possible to avoid a solution of the eigenvalue problem altogether. In this regard, the authors of this report are trying to develop the concept of "admissible vectors", which are large-dimensional vectors analogous to the admissible functions discussed earlier. Questions of definitions and convergence still remain, but the concept is sufficiently promising to warrant a sustained research effort.

9. Illustrative Example

Let us consider a specific structure consisting of a central substructure with a four-bladed rotor attached to it via a flexible

shaft. We shall take the central substructure in the form of a beam and assume that the rotor and shaft rotate uniformly relative to the beam about an axis perpendicular to it (see Fig. 2). The central beam, the rotor blades, and the shaft are assumed to be one-dimensional members. The shaft can be considered as substructure A and the entire rotor as substructure B so that the shaft and rotor can be considered as a type AB chain of substructures. Moreover, assuming that the rotor has more than two blades and that the rotor blades are identical, so that the rotor is symmetric when undeformed, the resulting equations of motion for the entire structure can be reduced to a set of constant-coefficient equations representing a gyroscopic system.

The actual derivation of the system kinetic energy is extremely involved and will not be presented here. Instead, we shall merely discuss the quantities describing the specific substructures which are needed in the kinematical considerations. A method for obtaining an explicit expression for the kinetic energy of the entire structure is discussed in the following sections.

First, to describe the motion of the central beam, let us consider the set of axes $x_C y_C z_C$ with the origin C at the mass center of the undeformed beam and with the x_C axis along the length of the beam. We shall take axes y_C and z_C to be perpendicular to the beam so as to form a triad in which the z_C axis is parallel to the shaft when the structure is undeformed. The orientation of $x_C y_C z_C$ relative to the inertial axes XYZ is described by three rotations: θ_{C1} about x_C, θ_{C2} about y_C, and θ_{C3} about z_C, in that order, so that the matrix T_{CO} of direction cosines between these axes can be written

$$T_{CO} = \left[\begin{array}{c|c|c} c\theta_{C2}c\theta_{C3} & s\theta_{C1}s\theta_{C2}c\theta_{C3} + c\theta_{C1}s\theta_{C3} & -c\theta_{C1}s\theta_{C2}c\theta_{C3} + s\theta_{C1}s\theta_{C3} \\ -c\theta_{C2}s\theta_{C3} & -s\theta_{C1}s\theta_{C2}s\theta_{C3} + c\theta_{C1}c\theta_{C3} & c\theta_{C1}s\theta_{C2}s\theta_{C3} + s\theta_{C1}c\theta_{C3} \\ s\theta_{C2} & -s\theta_{C1}c\theta_{C2} & c\theta_{C1}c\theta_{C2} \end{array}\right] \tag{38}$$

where $c\theta_{C1}$ and $s\theta_{C1}$ denote $\cos\theta_{C1}$ and $\sin\theta_{C1}$, respectively, etc.

Moreover, the radius vector from O to C, $\underset{\sim}{w}_{OC}$, in terms of components along axes XYZ is

$$\underset{\sim}{w}_{OC} = \{w_{OC_x} \quad w_{OC_y} \quad w_{OC_z}\}^T \tag{39}$$

and the angular velocity of axes $x_C y_C z_C$ in terms of components along these axes can be shown to be

$$\underset{\sim}{\Omega}_C = \begin{bmatrix} c\theta_{C2}c\theta_{C3} & s\theta_{C3} & 0 \\ -c\theta_{C2}s\theta_{C3} & c\theta_{C3} & 0 \\ s\theta_{C2} & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dot\theta_{C1} \\ \dot\theta_{C2} \\ \dot\theta_{C3} \end{Bmatrix} = \Theta_C \underset{\sim}{\dot\theta}_C \tag{40}$$

The position vector of any mass point in the one-dimensional beam is simply

$$\underset{\sim}{r}_C = \{x_C \quad 0 \quad 0\}^T \tag{41}$$

where x_C is the position of the point relative to C along the x_C axis and the elastic displacement of that point measured relative to $x_C y_C z_C$ is

$$\underset{\sim}{u}_C = \underset{\sim}{u}_C(x_C, t) = \{0 \quad v_C(x_C,t) \quad w_C(x_C,t)\}^T \tag{42}$$

Using Eqs. (38)-(42), we can obtain an explicit form of Eq. (1) for the absolute velocity of a mass point in the central beam. Such an explicit expression is not presented here for brevity.

Next, let us consider the shaft, which is a type A substructure. We shall take the shaft to be connected to the central beam at point A a distance r_{CA} from C along the x_C axis when the central beam is undeformed, positive in the $-x_C$ direction, and consider the axes $x_{CA}y_{CA}z_{CA}$ with origin at A and axis z_{CA} along the length of the shaft when it is undeformed. Assuming that axes $x_{CA}y_{CA}z_{CA}$ are parallel to axes $x_C y_C z_C$ when the central beam is undeformed, the matrix of direction cosines L_{GA} between these axes due to geometry alone is the identity matrix. Hence, substituting Eq. (42) into Eq. (4) and evaluating the result at $x_C =$ $-$ rCA, we can write the matrix of direction cosines L_{EA} L_{GA} between axes $x_{CA}y_{CA}z_{CA}$ and $x_C y_C z_C$ after deformation of the central beam as

$$L_{EA}\ L_{GA} = \begin{bmatrix} 1 & v_C'(-r_{CA},t) & w_C'(-r_{CA},t) \\ -v_C'(-r_{CA},t) & 1 & 0 \\ -w_C'(-r_{CA},t) & 0 & 1 \end{bmatrix} \tag{43}$$

where primes denote differentiations with respect to x_C. Similarly, we can write the angular velocity vector $\underset{\sim}{\dot\beta}_{CA}$ of axes $x_{CA}y_{CA}z_{CA}$ relative

to $x_C y_C z_C$ due to deformation in the form

$$\dot{\beta}_{CA} = \{0 \quad -\dot{w}_C'(-r_{CA},t) \quad \dot{v}_C'(-r_{CA},t)\}^T \tag{44}$$

Assuming that in equilibrium the shaft axes $x_A y_A z_A$ rotate relative to $x_{CA} y_{CA} z_{CA}$ with the uniform angular velocity ω_A about the z_{CA} axis, and that there is no oscillation of $x_A y_A z_A$ relative to $x_{CA} y_{CA} z_{CA}$, the angular velocity of $x_A y_A z_A$ relative to $x_{CA} y_{CA} z_{CA}$ is

$$\omega_A = \omega_A L_A L_{EA} L_{GA} T_{CO} e_3 \tag{45}$$

where $e_3 = \{0 \ 0 \ 1\}$ and the matrix of direction cosines L_A between $x_A y_A z_A$ and $x_{CA} y_{CA} z_{CA}$ is simply

$$L_A = \begin{bmatrix} \cos \omega_A t & \sin \omega_A t & 0 \\ -\sin \omega_A t & \cos \omega_A t & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{46}$$

In addition, the position vector of any mass point in the one-dimensional shaft can be written as

$$r_A = \{0 \quad 0 \quad z_A\}^T \tag{47}$$

and the elastic displacement of that point measured relative to axes $x_A y_A z_A$ has the form

$$u_A = u_A(z_A,t) = \{u_A(z_A,t) \quad v_A(z_A,t) \quad 0\}^T \tag{48}$$

Using Eqs. (38)-(48), we can obtain an explicit form of Eq. (7) for the absolute velocity of a mass point in the shaft. Again, such an expression is omitted here for brevity.

Finally, let us consider the entire rotor, whose geometric center is connected to the shaft at point B, a distance $r_{AB} = L_s$ from A along the z_A axis when the shaft is undeformed. Considering a set of axes $x_{AB} y_{AB} z_{AB}$ with origin at B and parallel to axes $x_A y_A z_A$ when the shaft is undeformed, the matrix of direction cosines L_{GB} between axes $x_A y_A z_A$ and $x_{AB} y_{AB} z_{AB}$ due to geometry is the identity matrix. Substituting Eq. (48) into Eq. (9), differentiating the result with respect to time, and evaluating the result at $z_A = L_s$ yields the angular velocity vector $\dot{\beta}_{AB}$ of axes $x_{AB} y_{AB} z_{AB}$ relative to axes $x_A y_A z_A$ due to the bending deformation. If, in addition, we allow the one-dimensional shaft to have torsional deformation, then the angular velocity vector $\dot{\beta}_{AB}$ has the explicit form

$$\dot{\beta}_{AB} = \{-\dot{v}_A'(L_s,t) \quad \dot{u}_A'(L_s,t) \quad \dot{\phi}(L_s,t)\}^T \tag{49}$$

where primes here denote differentiation with respect to z_A and $\phi(L_s,t)$ is the shaft torsional displacement at point B. Assuming

that each rotor blade is cantilevered to the shaft at point B, i.e., that axes $x_B y_B z_B$ do not rotate relative to axes $x_{AB} y_{AB} z_{AB}$, the angular velocity of $x_B y_B z_B$ relative to $x_{AB} y_{AB} z_{AB}$ is zero and the matrix of direction cosines L_B between these two sets of axes is the identity matrix. Hence, the matrix of direction cosines T_{BA} between axes $x_B y_B z_B$ and $x_A y_A z_A$ can be shown to be

$$T_{BA} = L_{EB} = \begin{bmatrix} 1 & \phi(L_s,t) & -u_A'(L_s,t) \\ -\phi(L_s,t) & 1 & -v_A'(L_s,t) \\ u_A'(L_s,t) & v_A'(L_s,t) & 1 \end{bmatrix} \tag{50}$$

To describe the position of a mass point in the rotor, we shall take one undeformed rotor blade to lie along the x_B axis, another to lie along the y_B axis, the third to lie along the $-x_B$ axis, and the fourth to lie along the $-y_B$ axis. Hence, we consider four one-dimensional members which are symmetric about the point B and have the domains of extension $0 \le x_B \le L_B$, $y_B = 0$; $0 \le y_B \le L_B$, $x_B = 0$; $-L_B \le x_B \le 0$, $y_B = 0$; $-L_B \le y_B \le 0$, $x_B = 0$. We can write the position of a mass point in the domains of extension $0 \le x_B \le L_B$, $y_B = 0$ and $-L_B \le x_B \le 0$, $y_B = 0$ as

$$\underset{\sim}{r}_B = \{x_B \quad 0 \quad 0\}^T \tag{51}$$

and the position of a mass point in the domains of extension $0 \le x_B \le L_B$, $y_B = 0$ and $-L_B \le x_B \le 0$, $y_B = 0$ as

$$\underset{\sim}{r}_B = \{0 \quad y_B \quad 0\}^T \tag{52}$$

The elastic deformations of a mass point in each domain of extension measured relative to axes $x_B y_B z_B$ can be written as

$$\begin{aligned}
\underset{\sim}{u}_{B1} &= \underset{\sim}{u}_{B1}(x_B,t) = \{0 \quad v_{B1}(x_B,t) \quad w_{B1}(x_B,t)\}^T, \ 0 \le x_B \le L_B,\ y_B = 0 \\
\underset{\sim}{u}_{B2} &= \underset{\sim}{u}_{B2}(y_B,t) = \{-v_{B2}(y_B,t) \quad 0 \quad w_{B2}(y_B,t)\}^T, \ 0 \le y_B \le L_B,\ x_B = 0 \\
\underset{\sim}{u}_{B3} &= \underset{\sim}{u}_{B3}(x_B,t) = \{0 \quad -v_{B3}(x_B,t) \quad w_{B3}(x_B,t)\}^T, \ -L_B \le x_B \le 0,\ y_B = 0 \\
\underset{\sim}{u}_{B4} &= \underset{\sim}{u}_{B4}(y_B,t) = \{v_{B4}(y_B,t) \quad 0 \quad w_{B4}(y_B,t)\}^T, \ -L_B \le y_B \le 0,\ x_B = 0
\end{aligned} \tag{53}$$

where $v_{Bi}(\zeta,t)$ $0 \le \zeta \le L_B$; $i = 1, 2, 3, 4$ are the elastic displacements of each member in the $x_B y_B z_B$ plane and $w_{Bi}(\zeta,t)$ $0 \le \zeta \le L_B$; $i = 1, 2, 3, 4$ are the elastic displacements of each member perpendicular to the $x_B y_B z_B$ plane. Note that ζ represents a local spatial coordinate for

each domain of extension. Using Eqs. (38)-(53), we can write an explicit form of Eq. (10) for the absolute velocity of any mass point in the rotor. As before, the resulting lengthy expression is omitted for brevity.

Turning our attention to the system potential energy, we note that for the particular example under consideration axes $x_By_Bz_B$ and $x_Ay_Az_A$ do not oscillate relative to axes $x_{AB}y_{AB}z_{AB}$ and $x_{CA}y_{CA}z_{CA}$, respectively, so that the function U in Eq. (14) does not appear in the potential energy. Moreover, assuming that the uniform angular velocity ω_A is large compared to all other angular velocity components, we can write Eq. (14) in the explicit form

$$V = \frac{1}{2}\int_{L_C} EI_C\left(\frac{\partial^2 \underset{\sim}{u}_C(x_C,t)}{\partial x_C^2}\right)^T\left(\frac{\partial^2 \underset{\sim}{u}_C(x_C,t)}{\partial x_C^2}\right)dx_C$$

$$+\int_o^{L_s} EI_A\left(\frac{\partial^2 \underset{\sim}{u}_A(z_A,t)}{\partial z_A^2}\right)^T\left(\frac{\partial^2 \underset{\sim}{u}_A(z_A,t)}{\partial z_A^2}\right)dz_A + \frac{1}{2}\int_o^{L_s} GJ_A\left(\frac{\partial\phi(z_A,t)}{\partial z_A}\right)^2 dz_A$$

$$+\frac{1}{2}\sum_{i=1}^{4}\int_o^{L_B}\left[EI_B\left(\frac{\partial^2 \underset{\sim}{u}_{Bi}(\zeta,t)}{\partial\zeta^2}\right)^T\left(\frac{\partial^2 \underset{\sim}{u}_{Bi}(\zeta,t)}{\partial\zeta^2}\right)\right.$$

$$\left.+\ \omega_A^2\rho_B(L_B^2-\zeta^2)\left(\frac{\partial \underset{\sim}{u}_{Bi}(\zeta,t)}{\partial\zeta}\right)^T\left(\frac{\partial \underset{\sim}{u}_{Bi}(\zeta,t)}{\partial\zeta}\right)\right]d\zeta \qquad (54)$$

where EI_C, EI_A, and EI_B are bending stiffnesses of the central beam, the shaft, and each rotor blade, respectively, GJ_A is the torsional stiffness of the shaft, and ρ_B is the mass per unit length of each rotor blade. It remains to specify the geometric boundary conditions for each substructure. To this end, we note that the mass center of the undeformed central beam always coincides with the origin of axes $x_Cy_Cz_C$ so that $\underset{\sim}{u}_C(0,t) = \underset{\sim}{0}$. In addition, we shall consider the shaft to be cantilevered to the central beam so that $\underset{\sim}{u}_A(0,t) = \frac{\partial \underset{\sim}{u}_A(z_A,t)}{\partial z_A}\Big|_{z_A=0} = \underset{\sim}{0}$ and $\phi(0,t) = 0$. Finally, since each rotor blade is cantilevered to the shaft, we have $\underset{\sim}{u}_{Bi}(0,t) = \frac{\partial \underset{\sim}{u}_{Bi}(\zeta,t)}{\partial\zeta}\Big|_{\zeta=0} = \underset{\sim}{0}$.

Equations (42), (48), and (53) give the elastic displacement of a mass point in each continuous substructure in terms of continuous functions of the spatial position and time. As indicated in Sec. 5, it is necessary to eliminate the spatial dependence. To this end,

let us introduce the specific discretization scheme for the example structure

$$\underset{\sim}{u}_C = \Phi_C \underset{\sim}{\eta}_C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \phi_{C1}(x_C) & \phi_{C2}(x_C) & 0 & 0 \\ 0 & 0 & \phi_{C1}(x_C) & \phi_{C2}(x_C) \end{bmatrix} \begin{Bmatrix} \eta_{C1}(t) \\ \eta_{C2}(t) \\ \vdots \\ \eta_{C4}(t) \end{Bmatrix} \quad (55a)$$

$$\underset{\sim}{u}_A = \Phi_A \underset{\sim}{\eta}_A = \begin{bmatrix} \phi_{A1}(z_A) & \phi_{A2}(z_A) & 0 & 0 \\ 0 & 0 & \phi_{A1}(z_A) & \phi_{A2}(z_A) \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \eta_{A1}(t) \\ \eta_{A2}(t) \\ \vdots \\ \eta_{A4}(t) \end{Bmatrix} \quad (55b)$$

$$\underset{\sim}{u}_{B1} = \Phi_{B1}\underset{\sim}{\eta}_B = \begin{bmatrix} 0 & 0 & 0 & 0 & & & \\ \phi_{B1}(\zeta) & \phi_{B2}(\zeta) & 0 & 0 & [0] & [0] & [0] \\ 0 & 0 & \phi_{B1}(\zeta) & \phi_{B2}(\zeta) & & & \end{bmatrix} \begin{Bmatrix} \eta_{B1}(t) \\ \eta_{B2}(t) \\ \vdots \\ \eta_{B16}(t) \end{Bmatrix}$$

$$\underset{\sim}{u}_{B2} = \Phi_{B2}\underset{\sim}{\eta}_B = \begin{bmatrix} & -\phi_{B1}(\zeta) & -\phi_{B2}(\zeta) & 0 & 0 & & \\ [0] & 0 & 0 & 0 & 0 & [0] & [0] \\ & 0 & 0 & \phi_{B1}(\zeta) & \phi_{B2}(\zeta) & & \end{bmatrix} \begin{Bmatrix} \eta_{B1}(t) \\ \eta_{B2}(t) \\ \vdots \\ \eta_{B16}(t) \end{Bmatrix}$$

$$(55c)$$

$$\underset{\sim}{u}_{B3} = \Phi_{B3}\underset{\sim}{\eta}_B = \begin{bmatrix} & & 0 & 0 & 0 & 0 & \\ [0] & [0] & -\phi_{B1}(\zeta) & -\phi_{B2}(\zeta) & 0 & 0 & [0] \\ & & 0 & 0 & \phi_{B1}(\zeta) & \phi_{B2}(\zeta) & \end{bmatrix} \begin{Bmatrix} \eta_{B1}(t) \\ \eta_{B2}(t) \\ \vdots \\ \eta_{B16}(t) \end{Bmatrix}$$

$$\underset{\sim}{u}_{B4} = \Phi_{B4}\underset{\sim}{\eta}_B = \begin{bmatrix} & & & \phi_{B1}(\zeta) & \phi_{B2}(\zeta) & 0 & 0 \\ [0] & [0] & [0] & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & \phi_{B1}(\zeta) & \phi_{B2}(\zeta) \end{bmatrix} \begin{Bmatrix} \eta_{B1}(t) \\ \eta_{B2}(t) \\ \vdots \\ \eta_{B16}(t) \end{Bmatrix}$$

where [0] denotes the 3×4 null matrix. Note that we have taken $n_C = 4$, $n_A = 4$, and $n_B = 16$ in Eqs. (54). In addition, to discretize the torsional displacement $\phi(z_A,t)$, we write

$$\phi(z_A,\ t) = \Theta_1(z_A)\eta_{\phi 1}(t) + \Theta_2(z_A)\eta_{\phi 2}(t)$$

$$= \{\Theta_1(z_A)\ \Theta_2(z_A)\}\{\eta_{\phi 1}(t)\ \eta_{\phi 2}(t)\}^T = \underset{\sim}{\Theta}^T \underset{\sim}{\eta}_\phi \quad (56)$$

so that the entire structure is represented by 6 + 4 + 4 + 2 + 16 = 32 degrees of freedom. Of course the space dependent functions appearing in Eqs. (55) must satisfy the geometric boundary conditions mentioned previously.

10. General Concepts for Symbolic Manipulation

The major obstacle encountered in the derivation of explicit equations of motion is the explicit evaluation of the system kinetic energy. Indeed, introducing Eqs. (1), (7), and (10) into Eq. (11), we obtain

$$
\begin{aligned}
T = {} & \frac{1}{2} m_C \dot{\underline{w}}_{OC}^T \dot{\underline{w}}_{OC} + \frac{1}{2} \dot{\underline{\Omega}}_C^T J_C \dot{\underline{\Omega}}_C + \frac{1}{2} \int_{m_C} \dot{\underline{u}}_C^T \dot{\underline{u}}_C \, dm_C + \dot{\underline{w}}_{OC}^T T_{CO}^T \bar{\dot{\underline{u}}}_C \\
& - \dot{\underline{w}}_{OC}^T T_{CO}^T \bar{\tilde{r}}_C \underline{\Omega}_C - \bar{\underline{H}}_C^T \underline{\Omega}_C + \frac{1}{2} m_A \dot{\underline{w}}_{CA}^T \dot{\underline{w}}_{CA} + \frac{1}{2} \underline{\Omega}_A^T J_A \underline{\Omega}_A \\
& + \frac{1}{2} \int_{m_A} \dot{\underline{u}}_A^T \dot{\underline{u}}_A \, dm_A + \dot{\underline{w}}_{CA}^T T_{AC}^T \bar{\dot{\underline{u}}}_A - \dot{\underline{w}}_{CA}^T T_{AC}^T \bar{\tilde{r}}_A \underline{\Omega}_A - \bar{\underline{H}}_A^T \underline{\Omega}_A \\
& + \frac{1}{2} m_B \dot{\underline{w}}_{AB}^T \dot{\underline{w}}_{AB} + \frac{1}{2} \underline{\Omega}_B^T J_B \underline{\Omega}_B + \frac{1}{2} \int_{m_B} \dot{\underline{u}}_B^T \dot{\underline{u}}_B \, dm_B + \dot{\underline{w}}_{AB}^T T_{BA}^T \bar{\dot{\underline{u}}}_B \\
& - \dot{\underline{w}}_{AB}^T T_{BA}^T \bar{\tilde{r}}_B \underline{\Omega}_B - \bar{\underline{H}}_B^T \underline{\Omega}_B
\end{aligned}
\tag{57}
$$

where

$$
\begin{aligned}
& m_C = \int_{m_C} dm_C \quad , \quad \bar{\tilde{r}}_C = \int_{m_C} (\tilde{r}_C + \tilde{u}_C) dm_C \\
& \bar{\dot{\underline{u}}}_C = \int_{m_C} \dot{\underline{u}}_C dm_C \quad , \quad J_C = \int_{m_C} (\tilde{r}_C + \tilde{u}_C)^T (\tilde{r}_C + \tilde{u}_C) dm_C \\
& \bar{\underline{H}}_C = \int_{m_C} (\tilde{r}_C + \tilde{u}_C)^T \dot{\underline{u}}_C dm_C
\end{aligned}
\tag{58}
$$

and similar definitions hold for m_A, m_B, $\bar{\tilde{r}}_A$, $\bar{\tilde{r}}_B$, $\bar{\dot{\underline{u}}}_A$, $\bar{\dot{\underline{u}}}_B$, J_A, J_B, H_A, and H_B. Upon contemplating the form of Eq. (57), one concludes that evaluating the system kinetic energy explicitly involves the calculation of lengthy matrix products. Moreover, these matrix products involve quantities of different degrees. Generally, higher degrees imply less significance and, in fact, many higher-degree terms can be ignored. Specifically, for the linearized problem discussed herein, terms that are higher than second degree in the generalized coordinates and generalized velocities are ignored. The complexity associated with the algebraic multiplication of many matrices and the high probability of human error in performing many such multiplications by hand makes it highly desirable to computerize these algebraic operations. A procedure whereby the

required algebraic operations can be implemented on a computer is outlined in this section.

Let us first point out that obtaining Eqs. (30) in explicit form is tantamount to determining the coefficient matrices M, F^T-F, and K_V-K_T which in turn are determined by specific knowledge of the matrices M, F, K_T, and K_V. The matrices M, F, and K_T can be obtained for a given structure by evaluating Eq. (57) explicitly in terms of the discretized coordinates, expanding the result in a Taylor series about the equilibrium configuration, applying a coordinate transformation to eliminate explicit time dependence, and then simply identifying the matrix elements as the coefficients of the various products of generalized coordinates and generalized velocities. Of course, the matrix K_V is specified for a given structure because it is assembled from the substructure stiffnesses.

The ideas used in implementing the required algebraic manipulation on a computer are best introduced via an explicit example. Let us consider the product

$$0.5\ m_A \dot{\underset{\sim}{w}}_{OC}^T T_{CO}^T \left[T_{CO} \dot{\underset{\sim}{w}}_{OC} - (\tilde{r}_{CA} + \tilde{u}_{CA}) \underset{\sim}{\Omega}_C + \dot{\underset{\sim}{u}}_{CA}\right] \tag{59}$$

which appears in the expansion of $\frac{1}{2}\ m_A \dot{\underset{\sim}{w}}_{CA}^T \dot{\underset{\sim}{w}}_{CA}$, where the latter was encountered in Eq. (57). The explicit calculation of this product is done in two steps. First, the resulting matrix expression is obtained by considering the vectors $\dot{\underset{\sim}{w}}_{OC}$, $\underset{\sim}{\Omega}_C$ and $\dot{\underset{\sim}{u}}_{CA}$ and the matrices T_{CO}, $\tilde{r}_{CA}$ and $\tilde{u}_{CA}$ symbolically, and then, knowing the elements of each matrix or vector explicitly for a specific structure, the matrix expression previously obtained is evaluated in terms of the explicit matrix elements. By separating the calculation into two steps, it is possible to ignore certain matrix products based on the minimum degree associated with each matrix in the product without having to evaluate the matrix product itself explicitly. In addition, sometimes the result of evaluating a matrix product is known apriori. In this case, each occurrence of the matrix product in a matrix expression can be recognized immediately and the suitable substitution made. Hence, simplification of the matrix expression is possible <u>before</u> it is evaluated explicitly, ultimately reducing the computation time required to obtain the explicit result.

For the moment, let us consider only those developments necessary for obtaining the matrix result. To calculate this product on the computer, we shall associate numbers with the different groups of characters that represent quantities to be manipulated algebraically, i.e., m_A, $\dot{\underset{\sim}{w}}_{OC}$, $\underset{\sim}{\Omega}_C$, $\dot{\underset{\sim}{u}}_{CA}$, T_{CO}, $\tilde{r}_{CA}$, and $\tilde{u}_{CA}$. These groups of characters constitute what will be referred to as symbols. Note that some symbols such as m_A are scalar symbols and others such as $\dot{\underset{\sim}{w}}_{OC}$, T_{CO}, etc., are matrix symbols. The association of numbers with symbols allows us to substitute manipulation of numbers for manipulation of symbols and is accomplished by the formation of a scalar symbol table and a matrix symbol table, Tables I and II. The scalar symbol table contains two entries per line. These entries give the character representation of a matrix symbol, the weight assigned to the matrix symbol, the dimensions N and M of the N×M matrix symbol, and a pointer which locates the algebraic representations of the individual matrix elements. This fifth column of the matrix symbol table containing the pointers will be discussed later. The weight of a scalar (matrix) symbol is determined by the analyst according to his knowledge of the scalar (matrix) symbol's magnitude or his desire to retain the symbol in the equations of motion. A high numerical value of the weight implies a less significant symbol. Note that here we assigned to the scalar m_A a weight of zero, to the 3×1 matrices $\dot{\underset{\sim}{w}}_{OC}$, $\underset{\sim}{\Omega}_C$, and $\dot{\underset{\sim}{u}}_{CA}$ a weight of unity, and to the 3×3 matrices T_{CO}, $\tilde{r}_{CA}$, and $\tilde{u}_{CA}$ weights of zero, zero, and unity, respectively. The number associated with a particular scalar or matrix symbol is the line number in the scalar or matrix symbol table. In this case, one is associated with m_A, 51 with $\dot{\underset{\sim}{w}}_{OC}$, 52 with $\underset{\sim}{\Omega}_C$, etc. Note that the first line in the matrix symbol table is numbered consecutively after the last line in the scalar symbol table.

Examining Eq. (59), we see that it is necessary to form algebraic products of symbols, for example 0.5 $m_A \dot{\underset{\sim}{w}}_{OC}^T T_{CO}^T$. To this end, we define terms. A term consists of a signed numerical coefficient, a pattern consisting of the numbers associated with each symbol in the product, and a weight representing the sum of the weights of each individual symbol appearing in the term. Note that the order of occurrance in a pattern of scalar symbols is arbitrary while the order of occurrence of matrix symbols must be preserved. Moreover, it is necessary to distinguish the tranpose of a matrix symbol from the matrix symbol itself.

This is accomplished by inserting in the pattern at the appropriate place the <u>negative</u> of the number associated with the particular matrix symbol. In view of the number-symbol associations and the weights of Tables I and II, we represent 0.5 $m_A \dot{w}_{\sim OC}^T T_{CO}^T$ as a term having a coefficient of 0.5, a pattern of 1, -51, -54 and a weight of 1. All terms, i.e. all coefficients, weights and patterns are stored in numbered storage stacks. The coefficient and weight of any term are always single numbers and are stored side-by-side in the <u>coefficients and weights free storage stack</u>, exhibited in the form of Table III. On the other hand, the pattern of a term may differ from the pattern of other terms and must be able to represent the product of any number of symbols. Because of the different lengths of different patterns, all patterns are stored in the separate <u>patterns free storage stack</u>, labeled as Table IV. Note that the coefficient and weight of 0.5 $m_A \dot{w}_{\sim OC}^T T_{CO}^T$ are stored in line 3 of Table III and the pattern is stored in lines 5, 6 and 7 of Table IV.

It is also necessary to form algebraic sums of terms, such as $T_{CO} \dot{w}_{\sim OC} - \tilde{r}_{CA} \Omega_{\sim C} - \tilde{u}_{CA} \Omega_{\sim C} + \dot{u}_{\sim CA}$, which we shall call <u>series</u>. We consider first only <u>matrix series</u>. Each matrix series is given a distinct <u>series name</u>. As examples, we shall call the matrix series consisting of the single term 0.5 $m_A \dot{w}_{\sim OC}^T T_{CO}^T$ by the name X, the matrix series $T_{CO} \dot{w}_{\sim OC} - \tilde{r}_{CA} \Omega_{\sim C} - \tilde{u}_{CA} \Omega_{\sim C} + u_{\sim CA}$ by the name Y, and the matrix series resulting from the product of X and Y by the name Z. A matrix series is described by a sequence of terms with coefficients and weights stored sequentially in the coefficients and weights free storage stack and with patterns stored sequentially in the patterns free storage stack. The summation of terms in the sequence is understood. Hence, the matrix series Y is described by the coefficients and weights stored in lines 5, 6, 7 and 8 of Table III and the patterns stored in lines 11-12, 14-15, 17-18 and 20 of Table IV. To distinguish sequences of terms forming matrix series, each matrix series is assigned a line in the <u>matrix series definition table</u>, Table V. Each line of this table contains seven entries giving: (1) the matrix series name, (2) the line number of the coefficient and weight in the coefficients and weights free storage stack of the first term in the matrix series, (3) the number of terms in the matrix series, (4) the line number of the beginning of the pattern in the patterns free storage stack of the first

term in the matrix series, (5) and (6) the dimensions N and M of the N×M matrix series, and (7) a pointer which locates the algebraic representations of the individual matrix elements which would result if the matrix series were to be expanded explicitly. The seventh column in the matrix series definition table containing the pointers will be discussed later. Assigning line 3 in the matrix series definition table to the matrix series Y, the third line of Table V contains the entries 5 and 11 giving the storage locations of the matrix series, the entry 4 indicating that there are four terms in the series, and the entries 3 and 1 indicating that evaluation of the matrix series results in a three by one matrix.

Let us now consider the multiplication of two matrix series, namely, the multiplication of Y by X, which can be performed term by term. The product of two terms yields a new term. If the total weight of the new term, given by adding up the weights of the two terms in the product, is greater than a specified value, for example 2, then the new term is deleted. Otherwise, the coefficient of the new term is the product of coefficients and the pattern of the new term is the concatenation of the patterns of the two terms in the product, where the concatenation preserves the order of occurrence of matrix symbols. Multiplying the four terms in Y by the single term in X, the first two resulting new terms each has a total weight of 2, the third new term has a total weight of 3 and is deleted, and the fourth new term has a total weight of 2. The first new term has a coefficient of 0.5 and a pattern of 1, -51, -54, 54, 51. The second new term has a coefficient of -0.5 and a pattern of 1, -51, -54, 55, 52. The third new term is deleted and the fourth new term has a coefficient of 0.5 and a pattern of 1, -51, -54, 53. Assigning line 6 of Table V to the matrix series Z, the resulting three terms retained in the product of Y by X are stored according to the information in line 6 of Table V.

Turning our attention to the additional developments necessary for evaluating the matrix series obtained above explicitly (in terms of the algebraic representations of the individual matrix elements) and for storing the result, let us consider only the last term in the matrix series Z, namely

$$0.5\ m_A \dot{\underset{\sim}{w}}_{OC}^T T_{CO}^T \dot{\underset{\sim}{u}}_{CA} \tag{60}$$

and discuss a matrix series Z1 formed from this single term and stored according to the information in line 7 of Table V. To evaluate Z1, Eq. (60), explicitly, the 3×1 matrices $\dot{\underset{\sim}{w}}_{OC}$ and $\dot{\underset{\sim}{u}}_{CA}$ and the 3×3 matrix T_{CO} must be known explicitly. To this end, we shall use

$$\dot{\underset{\sim}{w}}_{OC} = \begin{Bmatrix} \dot{w}_{OCx} \\ \dot{w}_{OCy} \\ \dot{w}_{OCz} \end{Bmatrix},\ \dot{\underset{\sim}{u}}_{CA} = \begin{Bmatrix} 0 \\ \phi_{CA1}\dot{\eta}_{C1}+\phi_{CA2}\dot{\eta}_{C2} \\ \phi_{CA1}\dot{\eta}_{C3}+\phi_{CA2}\dot{\eta}_{C4} \end{Bmatrix},\ T_{CO} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{61}$$

where the elements of these three matrix symbols represent <u>scalar series</u> and ϕ_{CA1}, ϕ_{CA2}, $\dot{\eta}_{C1}$, $\dot{\eta}_{C2}$, $\dot{\eta}_{C3}$, $\dot{\eta}_{C4}$, $\dot{w}_{OCx}$, $\dot{w}_{OCy}$ and $\dot{w}_{OCz}$ are scalar symbols with weights as defined in lines 2-10 of Table I. Scalar series are algebraic sums of terms with the pattern of each term containing only scalar symbols. Like matrix series, a scalar series is also described by a sequence of terms with coefficients and weights stored sequentially in the coefficients and weights free storage stack and with patterns stored sequentially in the patterns free storage stack. Again, the summation of terms in the sequence is understood. To distinguish sequences of terms forming scalar series, each scalar series is assigned a line in the <u>scalar series definition table</u>, Table VI. Each line of this table contains three entries giving the line number of the coefficient and weight in the coefficients and weights free storage stack of the first term in the scalar series, the number of terms in the series, and the line number of the beginning of the pattern in the patterns free storage stack of the first term in the scalar series. Note that no series name is associated with a scalar series. This is because scalar series are always algebraic representations of particular matrix elements and, as such, they are directly associated with either a matrix symbol or a matrix series which already has a series name. Considering the 3x1 matrix symbol $\dot{\underset{\sim}{w}}_{OC}$ and its explicit definition given in Eq. (61), the scalar series representing the (1, 1), (2, 1) and (3, 1) matrix elements are stored according to the information found in lines 1, 2 and 3 of the scalar series definition table. Moreover, the association between these three lines in Table VI and the matrix symbol $\dot{\underset{\sim}{w}}_{OC}$ is accomplished by entering 1, giving the location in Table VI of the (1, 1) matrix element, in column 5 of line 51 of the matrix symbol table, Table II.

If the matrix symbol in question has more than one column, then the information defining the scalar series representations of the matrix elements is stored in column-order in Table VI, i.e., one column after another. For example, the scalar series representing the (1, 1), (2, 1), (3, 1), (1, 2), (2, 2), (3, 2), (1, 3), (2, 3) and (3, 3) matrix elements of the 3×3 matrix symbol T_{CO} are stored according to the information found in lines 4, 5, 6, 7, 8, 9, 10, 11 and 12, respectively, of Table VI. Note the 4 appearing in column 5, line 54, of Table II which locates the (1, 1) matrix element of T_{CO}.

Now that the elements of the matrices $\dot{\underset{\sim}{w}}_{OC}$, $\dot{\underset{\sim}{u}}_{CA}$ and T_{CO} are scalar series defined explicitly and associated with the appropriate matrix symbols through column five of the matrix symbol table, the explicit evaluation of the matrix series Z1 can be performed according to the usual rules governing matrix multiplication. The evaluation requires simply the multiplication and addition of scalar series. The multiplication of scalar series is exactly analogous to the multiplication of matrix series discussed earlier and is performed term by term, where again any new terms with a total weight greater than 2 are deleted. The result of the explicit evaluation of Z1 is a scalar series representing the (1, 1) element of the resulting 1x1 matrix. This scalar series is stored according to the information found in line 16 of the scalar series definition table and is associated with the matrix series Z1 by entering 16 in column 7, line 7, of Table V.

11. <u>The Computer Program</u>

The ideas outlined in the preceeding section are appealing because of their simplicity and can be programmed easily in FORTRAN. Indeed, a FORTRAN computer program implementing these ideas has been written. The program is organized into subroutines, with each subroutine performing a different operation required in producing the matrices M, F and K_T explicitly (such as multiplication of matrix series, multiplication and addition of scalar series, etc.). However, in order to produce these matrices explicitly for a given structure, it is necessary to formulate an algorithm which executes the appropriate subroutine at the appropriate time. It is expected that the algorithm will be slightly different for each specific structure. In order to facilitate the analyst's job in formulating and changing the algorithm for a structure, a simple interpreter has also been written (in FORTRAN). The interpreter

accepts as input any of a set of twelve simple instructions and, depending on the instruction, causes a particular subroutine or complicated sequence of subroutines to be executed. Two of the twelve instructions are used to declare a group of characters to be either a scalar symbol with a weight or a matrix symbol with a weight and dimensions. Four instructions are used to input and output either a matrix series or the scalar series which define a matrix symbol or matrix series explicitly. One instruction is used to set the maximum weight for retaining terms. Another instruction is used to substitute a matrix series for every occurrence of a particular matrix symbol in another matrix series. Still another instruction is used to cause a matrix series to be evaluated in terms of the explicit matrix elements for each matrix symbol. Two instructions are used to define the vector of generalized coordinates and the vector of generalized velocities in terms of the individual scalar symbols representing each generalized coordinate and generalized velocity. And finally, one instruction is used to generate the coefficient matrices M, F and K_T from the scalar series representing the system kinetic energy in explicit form.

12. Example Application

Let us consider the specific structure discussed in Sec. 9 consisting of a central beam with a four-bladed rotor attached to it via a flexible shaft. The specific quantities needed in the kinematical procedure for this structure are presented in Sec. 9. In this section, however, for brevity of presentation, we shall represent each continuous coordinate by only one degree of freedom. This is tantamount to substituting $\phi_{C2}(x_C)=\phi_{A2}(z_A)=\phi_{B2}(\xi)=0$ in Eqs. (55) and $\Theta_2(z_A)=0$ in Eq. (56). Hence, the entire structure is represented by 6+2+2+1+8=19 degrees of freedom and the 19-dimensional configuration vector is given by

$$\underset{\sim}{q} = \{w_{OCx}, w_{OCy}, w_{OCz}, \theta_{C1}, \theta_{C2}, \theta_{C3}, \eta_{C1}, \eta_{C3}, \eta_{A1}, \eta_{A3}, \eta_{\phi 1},$$

$$\eta_{B1}, \eta_{B3}, \eta_{B5}, \eta_{B7}, \eta_{B9}, \eta_{B11}, \eta_{B13}, \eta_{B15}\}^T \tag{62}$$

To derive explicit equations for small motions about equilibrium, the developments of Sec. 8 are used to calculate the mass integrals, Eqs. (58). These integrals are presented for each substructure in Appendix A. Using the computer program and these mass integrals in conjunction with the kinematical relationships given in Sec. 8, the system kinetic energy, Eq. (57), has been evaluated explicitly and found to depend on

time. In order to eliminate the explicit time dependence, we substitute the coordinate transformations

$$\begin{aligned} \eta_{A1} &= \eta^*_{A1}\cos\omega_A t + \eta^*_{A3}\sin\omega_A t \\ \eta_{A3} &= -\eta^*_{A1}\sin\omega_A t + \eta^*_{A3}\cos\omega_A t \end{aligned} \tag{63a}$$

$$\begin{aligned} \eta_{B1} &= \eta^*_{B1} - \eta^*_{B5} + \eta^*_{B9}\cos\omega_A t + \eta^*_{B13}\sin\omega_A t \\ \eta_{B3} &= \eta^*_{B3} - \eta^*_{B7} + \eta^*_{B11}\cos\omega_A t + \eta^*_{B15}\sin\omega_A t \\ \eta_{B5} &= \eta^*_{B1} + \eta^*_{B5} - \eta^*_{B9}\sin\omega_A t + \eta^*_{B13}\cos\omega_A t \\ \eta_{B7} &= \eta^*_{B3} + \eta^*_{B7} - \eta^*_{B11}\sin\omega_A t + \eta^*_{B15}\cos\omega_A t \\ \eta_{B9} &= \eta^*_{B1} - \eta^*_{B5} - \eta^*_{B9}\cos\omega_A t - \eta^*_{B13}\sin\omega_A t \\ \eta_{B11} &= \eta^*_{B3} - \eta^*_{B7} - \eta^*_{B11}\cos\omega_A t - \eta^*_{B15}\sin\omega_A t \\ \eta_{B13} &= \eta^*_{B1} + \eta^*_{B5} + \eta^*_{B9}\sin\omega_A t - \eta^*_{B13}\cos\omega_A t \\ \eta_{B15} &= \eta^*_{B3} + \eta^*_{B7} + \eta^*_{B11}\sin\omega_A t - \eta^*_{B15}\cos\omega_A t \end{aligned} \tag{63b}$$

into the mass integrals of Appendix A. The resulting mass integrals are presented in Appendix B. Note that Eqs. (63b) represent a transformation to multiblade coordinates (see Ref. 5). The computer program is now used to evaluate the system kinetic energy explicitly in terms of the transformed coordinates and then to identify the coefficient matrices M, F and K_T. The constant matrices M, F and K_T along with the constant matrix K_V obtained from Eq. (54) by considering Eqs. (55) and (56) in conjunction with Eqs. (63) are presented element by element in Appendix C.

To obtain Eq. (30) from the matrices M, F, K_T and K_V found in Appendix C, it remains to consider the elimination of ignorable coordinates as discussed in Sec. 4. For the particular structure under consideration, the vector $\underset{\sim}{w}_{OC}$ and the rotational coordinate θ_{C3} are ignorable

coordinates. Regarding the conservation of the generalized momenta associated with these coordinates as constraints on the system, we can use these constraints to eliminate $\underset{\sim}{w}_{OC}$ and θ_{C3} from the formulation and, as a result, obtain the equatons of motion for the system in the form of Eq. (30). The constraint equations in question are presented in Appendix D.

13. Concluding Remarks

The purpose of this investigation is to develop a general approach to the dynamic synthesis of substructures consisting of an assemblage of point-connected flexible substructures. At least one of the substructures undergoes uniform rotation in the equilibrium state, so that the system is gyroscopic. The mathematical model is intended to permit the determination of the modal characteristics of helicopters.

The approach is based on the substructure synthesis concept, whereby the elastic motion of a flexible substructure is represented by a suitable set of "modes". These modes need not represent natural modes of the substructures (corresponding to given boundary conditions) but can consist of a set of so-called admissible functions. A stationarity principle and an ensuing inclusion principle for gyroscopic systems permit the assessment of the effect of truncating the set of admissible functions on the system natural frequencies. Some rational criteria for the selection of the admissible functions are provided.

The method is applied to a simplified model of a helicopter, consisting of a beam simulating the airframe, a transmission shaft, and four elastic blades simulating the rotor. Use of the method of multiblade coordinates permits the reduction of the system of equations with periodic coefficients to one with constant coefficients, where the latter is of gyroscopic type.

The enormous amount of algebraic operations involved in the derivation of the coefficient matrices makes automation a virtual necessity. To this end, a program for symbolic manipulation on a digital computer has been developed. The program has been used to derive the coefficient matrices for the specific example considered.

This investigation represents a first step toward a rational approach to a problem of extreme complexity. Many aspects of the problem

still require answers. One of these problems is the concept of admissible vectors. This would permit the representation of discrete substructures by a set of vectors which do not necessarily represent natural modes of the substructure. Another problem requiring attention is that of multipoint connections for flexible gyroscopic systems. Although this problem may not appear to be so critical in a helicopter, the capability of simulating fixed wings should be beneficial.

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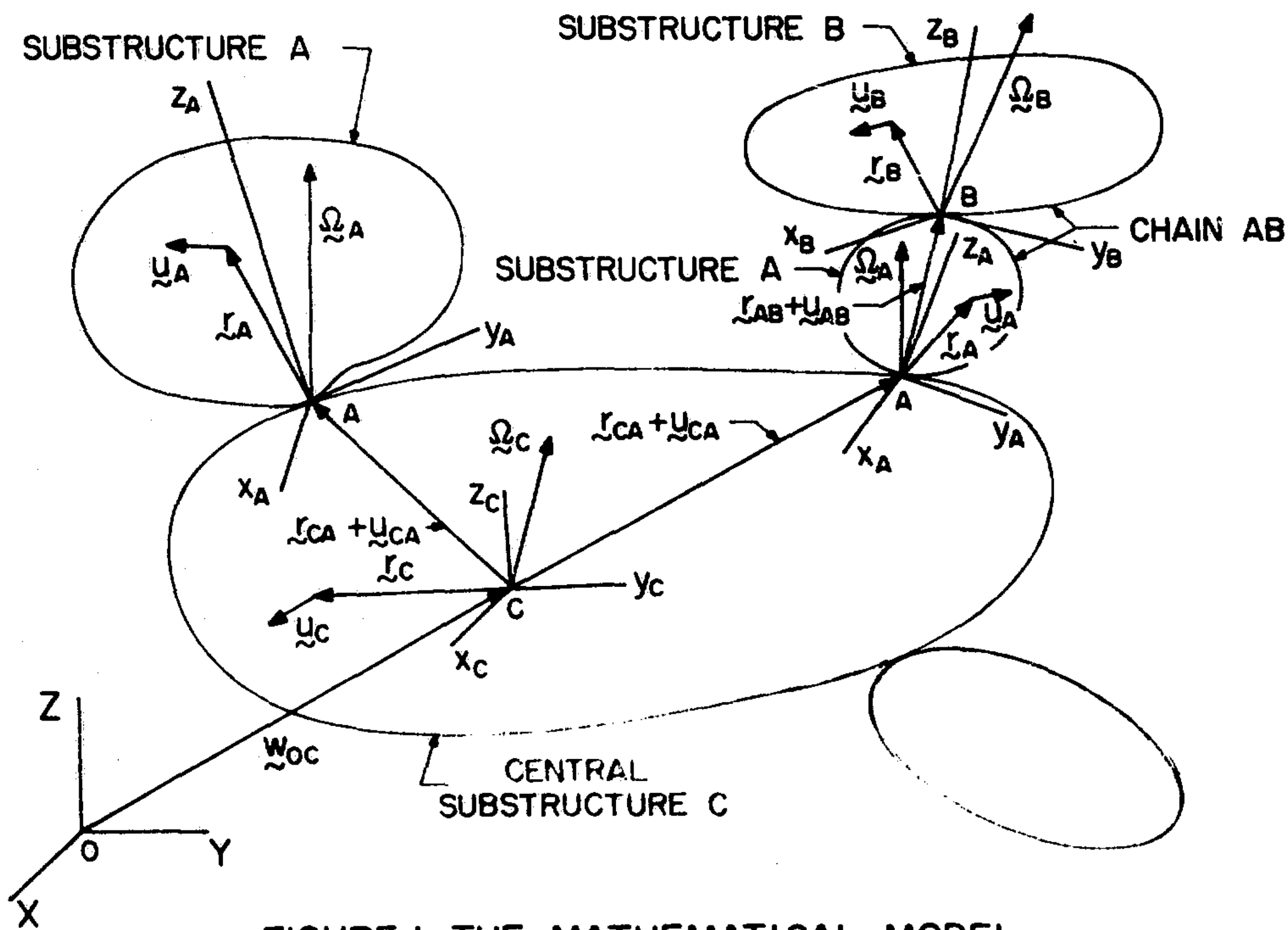

FIGURE I. THE MATHEMATICAL MODEL

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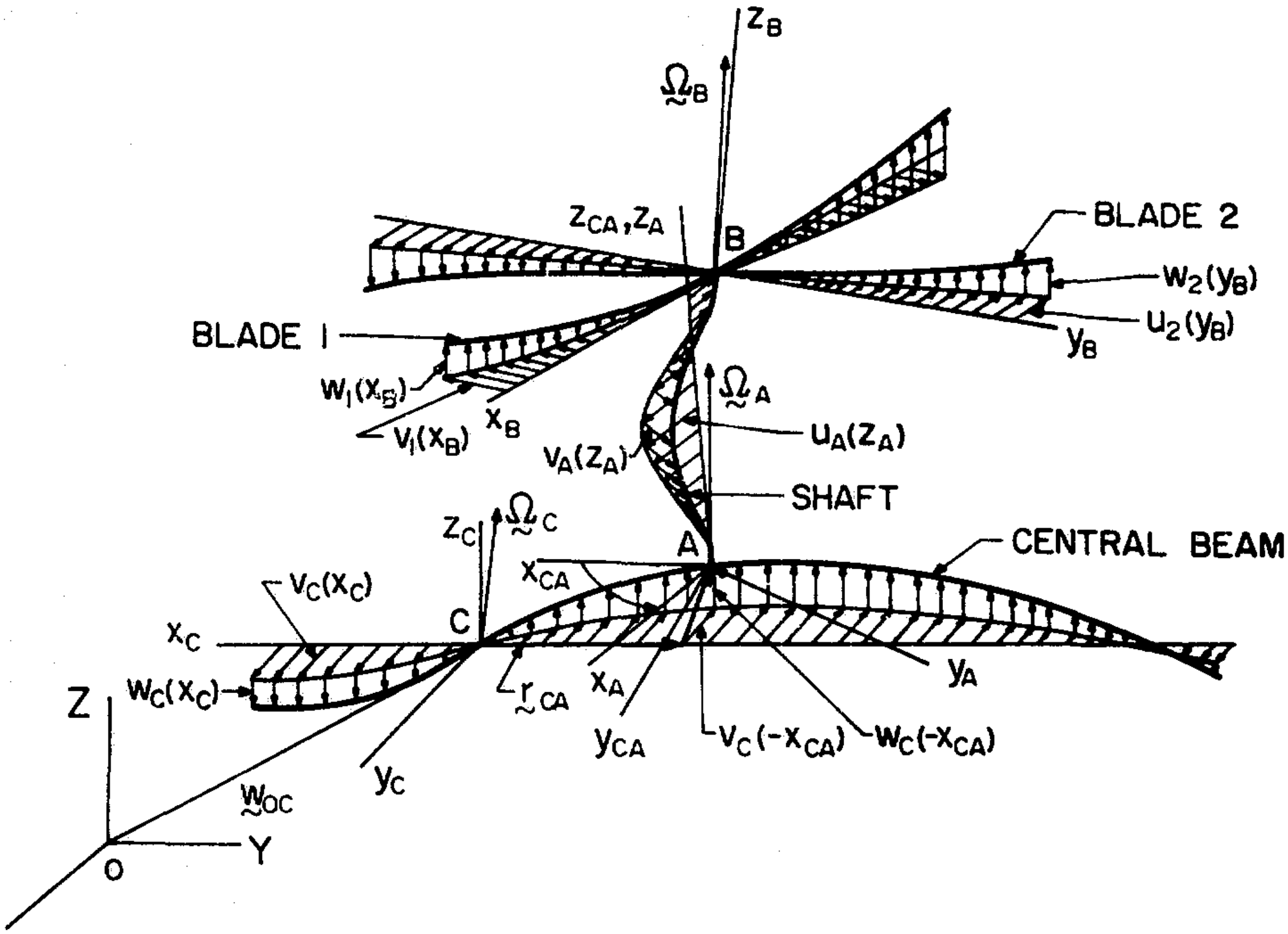

FIGURE 2. EXAMPLE STRUCTURE

	Symbol	Weight
1	m_A	0
2	ϕ_{CA1}	0
3	ϕ_{CA2}	0
4	$\dot{\eta}_{C1}$	1
5	$\dot{\eta}_{C2}$	1
6	$\dot{\eta}_{C3}$	1
7	$\dot{\eta}_{C4}$	1
8	$\dot{w}_{OCx}$	1
9	$\dot{w}_{OCy}$	1
10	$\dot{w}_{OCz}$	1
11	--	--
12	--	--
50	--	--

Table I - Scalar Symbol Table

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	Symbol	Weight	N	M	ISSDT
51	$\dot{\underset{\sim}{w}}_{OC}$	1	3	1	1
52	$\underset{\sim}{\Omega}_C$	1	3	1	--
53	$\underset{\sim}{u}_{CA}$	1	3	1	13
54	T_{CO}	0	3	3	4
55	$\tilde{r}_{CA}$	0	3	3	--
56	$\tilde{u}_{CA}$	1	3	3	--
57	--	--	--	--	--
100	--	--	--	--	--

Table II - Matrix Symbol Table

	Coefficient of term	Weight of term
1	0.0	0
2	--	--
3	0.5	1
4	--	--
5	1.0	1
6	-1.0	1
7	-1.0	2
8	1.0	1
9	--	--
10	0.5	2
11	0.5	2
12	-0.5	2
13	0.5	2
14	--	--
15	1.0	1
16	1.0	1
17	1.0	1

18	1.0	0
19	--	--
20	1.0	1
21	1.0	1
22	1.0	1
23	1.0	1
24	--	--
25	--	--
26	0.5	2
27	0.5	2
28	0.5	2
29	0.5	2
30	--	--
31	--	--
32	--	--
1000	--	--

Table III - Coefficients and Weights Free Storage Stack

1	32766	21	32766	41	32766	61	6	81	6
2	32767	22	32767	42	32767	62	32766	82	10
3	--	23	--	43	--	63	3	83	32766
4	--	24	--	44	8	64	7	84	1
5	1	25	1	45	32766	65	32766	85	3
6	-51	26	-51	46	32767	66	32767	86	7
7	-54	27	-54	47	9	67	--	87	10
8	32766	28	54	48	32766	68	--	88	32766
9	32767	29	51	49	32767	69	1	89	32767
10	--	30	32766	50	10	70	2	90	1
11	54	31	1	51	32766	71	4	91	-51
12	51	32	-51	52	32767	72	9	92	-54
13	32766	33	-54	53	2	73	32766	93	53
14	55	34	55	54	4	74	1	94	32766
15	52	35	52	55	32766	75	3	95	32767
16	32766	36	32766	56	3	76	5		
17	56	37	1	57	5	77	9	5000	--
18	52	38	-51	58	32766	78	32766		
19	32766	39	-54	59	32767	79	1		
20	53	40	53	60	2	80	2		

32766 denotes the end of a term

32767 denoted the end of a series

Table VI - Pattern Free Storage Stack

	Series Name	Loc.of Coeffs.	No.of Terms	Loc.of Patterns	N	M	ISSDT
1	-	-	-	-	-	-	-
2	-	-	-	-	-	-	-
3	Y	5	4	11	3	1	-
4	X	3	1	5	1	3	-
5	-	-	-	-	-	-	-
6	Z	11	3	25	1	1	-
7	Z1	10	1	90	1	1	16
50	-	-	-	-	-	-	-

Table V - Matrix Series Definition Table

	Loc.of Coeffs.	No.of Terms	Loc.of Patterns
1	15	1	44
2	16	1	47
3	17	1	50
4	18	1	1
5	1	1	1
6	1	1	1
7	1	1	1
8	18	1	1
9	1	1	1
10	1	1	1
11	1	1	1
12	18	1	1
13	1	1	1
14	20	2	53
15	22	2	60
16	26	4	69
100	-	-	-

Table VI - Scalar Series Definition Table

Appendix A

i. The mass integrals for substructure C

$$J_C = \begin{bmatrix} \bar{\phi}_{C11}(\eta_{C1}^2 + \eta_{C3}^2) & -\bar{\phi}_{xC1}\eta_{C1} & -\bar{\phi}_{xC1}\eta_{C3} \\ -\bar{\phi}_{xC1}\eta_{C1} & B_C + \bar{\phi}_{C11}\eta_{C3}^2 & -\bar{\phi}_{C11}\eta_{C1}\eta_{C3} \\ -\bar{\phi}_{xC1}\eta_{C3} & -\bar{\phi}_{C11}\eta_{C1}\eta_{C3} & B_C + \bar{\phi}_{C11}\eta_{C1}^2 \end{bmatrix} \tag{A1}$$

$$M_C = \bar{\phi}_{C11}(\dot{\eta}_{C1}^2 + \dot{\eta}_{C3}^2) \tag{A2}$$

$$\dot{\bar{U}}_C = \begin{Bmatrix} 0 \\ \bar{\phi}_{C1}\dot{\eta}_{C1} \\ \bar{\phi}_{C1}\dot{\eta}_{C3} \end{Bmatrix} \tag{A3}$$

$$\bar{\bar{r}}_C = \begin{bmatrix} 0 & -\bar{\phi}_{C1}\eta_{C3} & \bar{\phi}_{C1}\eta_{C1} \\ \bar{\phi}_{C1}\eta_{C3} & 0 & 0 \\ -\bar{\phi}_{C1}\eta_{C1} & 0 & 0 \end{bmatrix} \tag{A4}$$

$$\bar{H}_C = \begin{Bmatrix} \bar{\phi}_{C11}(\dot{\eta}_{C1}\eta_{C3} - \dot{\eta}_{C3}\eta_{C1}) \\ \bar{\phi}_{xC1}\dot{\eta}_{C3} \\ -\bar{\phi}_{xC1}\dot{\eta}_{C1} \end{Bmatrix} \tag{A5}$$

where we have used the definitions

$$\bar{\phi}_{C11} = \int_{m_C} \phi_{C1}^2(x_C)\,dm_C \quad , \quad \bar{\phi}_{xC1} = \int_{m_C} x_C\phi_{C1}(x_C)\,dm_C$$

$$\bar{\phi}_{C1} = \int_{m_C} \phi_{C1}(x_C)\,dm_C \quad , \quad B_C = \int_{m_C} x_C^2\,dm_C \tag{A6}$$

ii. The mass integrals for substructure A

$$J_A = \begin{bmatrix} B_A + \bar{\phi}_{A11}\eta_{A3}^2 & -\bar{\phi}_{A11}\eta_{A1}\eta_{A3} & -\bar{\phi}_{zA1}\eta_{A1} \\ -\bar{\phi}_{A11}\eta_{A1}\eta_{A3} & B_A + \bar{\phi}_{A11}\eta_{A1}^2 & -\bar{\phi}_{zA1}\eta_{A3} \\ -\bar{\phi}_{zA1}\eta_{A1} & -\bar{\phi}_{zA1}\eta_{A3} & \bar{\phi}_{A11}(\eta_{A1}^2+\eta_{A3}^2) \end{bmatrix} \tag{A7}$$

$$M_A = \bar{\phi}_{A11}(\dot{\eta}_{A1}^2 + \dot{\eta}_{A3}^2) \tag{A8}$$

$$\dot{\bar{u}}_A = \begin{Bmatrix} \bar{\phi}_{A1}\dot{\eta}_{A1} \\ \bar{\phi}_{A1}\dot{\eta}_{A3} \\ 0 \end{Bmatrix} \tag{A9}$$

$$\bar{\bar{r}}_A = \begin{bmatrix} 0 & -\frac{1}{2} m_A L_S & \bar{\phi}_{A1}\eta_{A3} \\ \frac{1}{2} m_A L_S & 0 & -\bar{\phi}_{A1}\eta_{A1} \\ -\bar{\phi}_{A1}\eta_{A3} & \bar{\phi}_{A1}\eta_{A1} & 0 \end{bmatrix} \tag{A10}$$

$$\bar{H}_A = \begin{Bmatrix} \bar{\phi}_{zA1}\dot{\eta}_{A3} \\ -\bar{\phi}_{zA1}\dot{\eta}_{A1} \\ \bar{\phi}_{A11}(\eta_{A3}\dot{\eta}_{A1} - \eta_{A1}\dot{\eta}_{A3}) \end{Bmatrix} \tag{A11}$$

where we have used the definitions

$$\begin{aligned} \bar{\phi}_{A11} &= \int_{m_A} \phi_{A1}^2(z_A)\,dm_A \quad , \quad \bar{\phi}_{zA1} = \int_{m_A} z_A\phi_{A1}(z_A)\,dm_A \\ \bar{\phi}_{A1} &= \int_{m_A} \phi_{A1}(z_A)\,dm_A \quad , \quad B_A = \int_{m_A} z_A^2\,dm_A \end{aligned} \tag{A12}$$

iii. <u>The mass integrals for substructure B</u>

$$J_B = \left[\begin{array}{c|c|c} 2B_B+\bar{\phi}_{B11}(\eta_{B1}^2+\eta_{B3}^2+\eta_{B7}^2+\eta_{B9}^2+\eta_{B11}^2+\eta_{B15}^2) & \bar{\phi}_{\zeta B1}(-\eta_{B1}+\eta_{B5}-\eta_{B9}+\eta_{B13}) & \bar{\phi}_{\zeta B1}(-\eta_{B3}+\eta_{B11}) + \bar{\phi}_{B11}(\eta_{B5}\eta_{B7}-\eta_{B13}\eta_{B15}) \\ \bar{\phi}_{\zeta B1}(-\eta_{B1}+\eta_{B5}-\eta_{B9}+\eta_{B13}) & 2B_B+\bar{\phi}_{B11}(\eta_{B3}^2+\eta_{B5}^2+\eta_{B7}^2+\eta_{B11}^2+\eta_{B13}^2+\eta_{B15}^2) & \bar{\phi}_{\zeta B1}(-\eta_{B7}+\eta_{B15}) + \bar{\phi}_{B11}(-\eta_{B1}\eta_{B3}+\eta_{B9}\eta_{B11}) \\ \bar{\phi}_{\zeta B1}(-\eta_{B3}+\eta_{B11}) + \bar{\phi}_{B11}(\eta_{B5}\eta_{B7}-\eta_{B13}\eta_{B15}) & \bar{\phi}_{\zeta B1}(-\eta_{B7}+\eta_{B15}) + \bar{\phi}_{B11}(-\eta_{B1}\eta_{B3}+\eta_{B9}\eta_{B11}) & 4B_B+\bar{\phi}_{B11}(\eta_{B1}^2+\eta_{B5}^2+\eta_{B9}^2+\eta_{B13}^2) \end{array}\right] \tag{A13}$$

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$$M_B = \bar{\phi}_{B11}(\dot{\eta}_{B1}^2+\dot{\eta}_{B3}^2+\dot{\eta}_{B5}^2+\dot{\eta}_{B7}^2+\dot{\eta}_{B9}^2+\dot{\eta}_{B11}^2+\dot{\eta}_{B13}^2+\dot{\eta}_{B15}^2) \tag{A14}$$

$$\bar{\dot{u}}_B = \begin{Bmatrix} \bar{\phi}_{B1}(-\dot{\eta}_{B5}+\dot{\eta}_{B13}) \\ \bar{\phi}_{B1}(\dot{\eta}_{B1}-\dot{\eta}_{B9}) \\ \bar{\phi}_{B1}(\dot{\eta}_{B3}+\dot{\eta}_{B7}+\dot{\eta}_{B11}+\dot{\eta}_{B15}) \end{Bmatrix} \tag{A15}$$

$$\bar{\bar{r}}_B = \begin{bmatrix} 0 & -\bar{\phi}_{B1}(\eta_{B3}+\eta_{B7}+\eta_{B11}+\eta_{B15}) & -\bar{\phi}_{B1}(-\eta_{B1}+\eta_{B9}) \\ \bar{\phi}_{B1}(\eta_{B3}+\eta_{B7}+\eta_{B11}+\eta_{B15}) & 0 & -\bar{\phi}_{B1}(-\eta_{B5}+\eta_{B13}) \\ \bar{\phi}_{B1}(-\eta_{B1}+\eta_{B9}) & \bar{\phi}_{B1}(-\eta_{B5}+\eta_{B13}) & 0 \end{bmatrix} \tag{A16}$$

$$\bar{H}_B = \begin{Bmatrix} \bar{\phi}_{B11}(\dot{\eta}_{B1}\eta_{B3}-\dot{\eta}_{B3}\eta_{B1}-\dot{\eta}_{B9}\eta_{B11}+\dot{\eta}_{B11}\eta_{B9}) + \bar{\phi}_{\zeta B1}(-\dot{\eta}_{B7}+\dot{\eta}_{B15}) \\ \bar{\phi}_{B11}(\dot{\eta}_{B5}\eta_{B7}-\dot{\eta}_{B7}\eta_{B5}-\dot{\eta}_{B13}\eta_{B15}+\dot{\eta}_{B15}\eta_{B13}) + \bar{\phi}_{\zeta B1}(\dot{\eta}_{B3}-\dot{\eta}_{B11}) \\ \bar{\phi}_{\zeta B1}(-\dot{\eta}_{B1}-\dot{\eta}_{B5}-\dot{\eta}_{B9}-\dot{\eta}_{B13}) \end{Bmatrix} \tag{A17}$$

where we have used the definitions

$$\bar{\phi}_{B11} = \int_0^{L_B} \rho_B \phi_{B1}^2(\zeta)d\zeta \qquad \bar{\phi}_{\zeta B1} = \int_0^{L_B} \rho_B \zeta \phi_{B1}(\zeta)d\zeta$$

$$\bar{\phi}_{B1} = \int_0^{L_B} \rho_B \phi_{B1}(\zeta)d\zeta \qquad B_B = \int_0^{L_B} \rho_B \zeta^2 d\zeta \tag{A18}$$

Appendix B

i. The mass integrals for substructure A

$$
J_A = \begin{bmatrix}
B_A+\bar{\phi}_{A11}(\overset{*2}{\eta}_{A1}s^2\omega_A t+\overset{*2}{\eta}_{A3}c^2\omega_A t-2\overset{*}{\eta}_{A1}\overset{*}{\eta}_{A3}s\omega_A\ c\omega_A t) & -\bar{\phi}_{A11}[s\omega_A tc\omega_A t(\overset{*2}{\eta}_{A1}-\overset{*2}{\eta}_{A3})+\overset{*}{\eta}_{A1}\overset{*}{\eta}_{A3}(s^2\omega_A t-c^2\omega_A t)] & -\bar{\phi}_{zA1}(\overset{*}{\eta}_{A1}c\omega_A t+\overset{*}{\eta}_{A3}s\omega_A t) \\
\bar{\phi}_{A11}[s\omega_A tc\omega_A t(\overset{*2}{\eta}_{A1}-\overset{*2}{\eta}_{A3}) + \overset{*}{\eta}_{A1}\overset{*}{\eta}_{A3}(s^2\omega_A t-c^2\omega_A t)] & B_A+\bar{\phi}_{A11}(\overset{*2}{\eta}_{A1}c^2\omega_A t+\overset{*2}{\eta}_{A3}s^2\omega_A t+2\overset{*}{\eta}_{A1}\overset{*}{\eta}_{A3}s\omega_A tc\omega_A t) & \bar{\phi}_{zA1}(\overset{*}{\eta}_{A1}s\omega_A t-\overset{*}{\eta}_{A3}c\omega_A t) \\
-\bar{\phi}_{zA1}(\overset{*}{\eta}_{A1}c\omega_A t+\overset{*}{\eta}_{A3}s\omega_A t) & \bar{\phi}_{zA1}(\overset{*}{\eta}_{A1}s\omega_A t-\overset{*}{\eta}_{A3}c\omega_A t) & \bar{\phi}_{A11}(\overset{*2}{\eta}_{A1}+\overset{*2}{\eta}_{A3})
\end{bmatrix} \tag{B1}
$$

$$
M_A = \bar{\phi}_{A11}[\dot{\overset{*2}{\eta}}_{A1}+\dot{\overset{*2}{\eta}}_{A3}+\omega_A^2(\overset{*2}{\eta}_{A1}+\overset{*2}{\eta}_{A3}) + 2\omega_A(\dot{\overset{*}{\eta}}_{A1}\overset{*}{\eta}_{A3}-\dot{\overset{*}{\eta}}_{A3}\overset{*}{\eta}_{A1})] \tag{B2}
$$

$$
\bar{\dot{\underset{\sim}{u}}}_A = \begin{Bmatrix}
\bar{\phi}_{A1}[\dot{\overset{*}{\eta}}_{A1}c\omega_A t+\dot{\overset{*}{\eta}}_{A3}s\omega_A t+\omega_A(-\overset{*}{\eta}_{A1}s\omega_A t+\overset{*}{\eta}_{A3}c\omega_A t)] \\
\bar{\phi}_{A1}[-\dot{\overset{*}{\eta}}_{A1}s\omega_A t+\dot{\overset{*}{\eta}}_{A3}c\omega_A t-\omega_A(\overset{*}{\eta}_{A1}c\omega_A t+\overset{*}{\eta}_{A3}s\omega_A t)] \\
0
\end{Bmatrix} \tag{B3}
$$

$$
\bar{\tilde{r}}_A = \begin{bmatrix}
0 & -\frac{1}{2}m_A L_S & \bar{\phi}_{A1}(-\overset{*}{\eta}_{A1}s\omega_A t+\eta_{A3}c\omega_A t) \\
\frac{1}{2}m_A L_S & 0 & -\bar{\phi}_{A1}(\overset{*}{\eta}_{A1}c\omega_A t+\overset{*}{\eta}_{A3}s\omega_A t) \\
\bar{\phi}_{A1}(\overset{*}{\eta}_{A1}s\omega_A t-\overset{*}{\eta}_{A3}c\omega_A t) & \bar{\phi}_{A1}(\overset{*}{\eta}_{A1}c\omega_A t+\overset{*}{\eta}_{A3}s\omega_A t) & 0
\end{bmatrix} \tag{B4}
$$

$$
\bar{\underset{\sim}{H}}_A = \begin{Bmatrix}
\bar{\phi}_{zA1}[-\dot{\overset{*}{\eta}}_{A1}s\omega_A t+\dot{\overset{*}{\eta}}_{A3}c\omega_A t-\omega_A(\overset{*}{\eta}_{A1}c\omega_A t+\overset{*}{\eta}_{A3}s\omega_A t)] \\
\bar{\phi}_{zA1}[-\dot{\overset{*}{\eta}}_{A1}c\omega_A t-\dot{\overset{*}{\eta}}_{A3}s\omega_A t+\omega_A(\overset{*}{\eta}_{A1}s\omega_A t-\overset{*}{\eta}_{A3}c\omega_A t)] \\
\bar{\phi}_{A11}[\overset{*}{\eta}_{A3}\dot{\overset{*}{\eta}}_{A1}-\overset{*}{\eta}_{A1}\dot{\overset{*}{\eta}}_{A3}+\omega_A(\overset{*2}{\eta}_{A1}+\overset{*2}{\eta}_{A3})]
\end{Bmatrix} \tag{B5}
$$

where $\bar{\phi}_{A11}$, $\bar{\phi}_{zA1}$, $\bar{\phi}_{A1}$ and B_A were defined in Appendix A.

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ii. <u>The mass integrals for substructure B</u>

The elements of J_B are:

$$J_{B11} = 2B_B+2\bar{\phi}_{B11}[\overset{*}{\eta}{}_{B1}^{2}+2\overset{*}{\eta}{}_{B3}^{2}+\overset{*}{\eta}{}_{B5}^{2}+2\overset{*}{\eta}{}_{B7}^{2}+\overset{*}{\eta}{}_{B9}^{2}c^2\omega_A t+\overset{*}{\eta}{}_{B11}^{2}+\overset{*}{\eta}{}_{B13}^{2}s^2\omega_A t+\overset{*}{\eta}{}_{B15}^{2}$$
$$-2\overset{*}{\eta}_{B1}\overset{*}{\eta}_{B5}+2\overset{*}{\eta}_{B9}\overset{*}{\eta}_{B13}s\omega_A t c\omega_A t]$$

$$J_{B12} = 4\bar{\phi}_{zB1}\overset{*}{\eta}_{B5}$$

$$J_{B13} = -2\bar{\phi}_{zB1}(\overset{*}{\eta}_{B11}c\omega_A t+\overset{*}{\eta}_{B15}s\omega_A t)+2\bar{\phi}_{B11}[-\overset{*}{\eta}_{B3}\overset{*}{\eta}_{B9}s\omega_A t+\overset{*}{\eta}_{B3}\overset{*}{\eta}_{B13}c\omega_A t$$
$$-\overset{*}{\eta}_{B7}\overset{*}{\eta}_{B9}s\omega_A t+\overset{*}{\eta}_{B7}\overset{*}{\eta}_{B13}c\omega_A t-\overset{*}{\eta}_{B1}\overset{*}{\eta}_{B11}s\omega_A t-\overset{*}{\eta}_{B5}\overset{*}{\eta}_{B11}s\omega_A t+\overset{*}{\eta}_{B1}\overset{*}{\eta}_{B15}c\omega_A t$$
$$+\overset{*}{\eta}_{B5}\overset{*}{\eta}_{B15}c\omega_A t]$$

$$J_{B21} = 4\bar{\phi}_{zB1}\overset{*}{\eta}_{B5}$$

$$J_{B22} = 2B_B+2\bar{\phi}_{B11}[\overset{*}{\eta}{}_{B1}^{2}+2\overset{*}{\eta}{}_{B3}^{2}+\overset{*}{\eta}{}_{B5}^{2}+2\overset{*}{\eta}{}_{B7}^{2}+\overset{*}{\eta}{}_{B9}^{2}s^2\omega_A t+\overset{*}{\eta}{}_{B11}^{2}+\overset{*}{\eta}{}_{B13}^{2}c^2\omega_A t+\overset{*}{\eta}{}_{B15}^{2}+2\overset{*}{\eta}_{B1}\overset{*}{\eta}_{B5}$$
$$-2\overset{*}{\eta}_{B9}\overset{*}{\eta}_{B13}s\omega_A t\ c\omega_A t]$$

$$J_{B23} = 2\bar{\phi}_{zB1}(\overset{*}{\eta}_{B11}s\omega_A t-\overset{*}{\eta}_{B15}c\omega_A t)+2\bar{\phi}_{B11}[-\overset{*}{\eta}_{B3}\overset{*}{\eta}_{B9}c\omega_A t-\overset{*}{\eta}_{B3}\overset{*}{\eta}_{B13}s\omega_A t+\overset{*}{\eta}_{B7}\overset{*}{\eta}_{B9}c\omega_A t$$
$$+\overset{*}{\eta}_{B7}\overset{*}{\eta}_{B13}s\omega_A t-\overset{*}{\eta}_{B1}\overset{*}{\eta}_{B11}c\omega_A t+\overset{*}{\eta}_{B5}\overset{*}{\eta}_{B11}c\omega_A t-\overset{*}{\eta}_{B1}\overset{*}{\eta}_{B15}s\omega_A t+\overset{*}{\eta}_{B5}\overset{*}{\eta}_{B15}s\omega_A t]$$

$$J_{B31} = -2\bar{\phi}_{zB1}(\overset{*}{\eta}_{B11}c\omega_A t+\overset{*}{\eta}_{B15}s\omega_A t)+2\bar{\phi}_{B11}[-\overset{*}{\eta}_{B3}\overset{*}{\eta}_{B9}s\omega_A t+\overset{*}{\eta}_{B3}\overset{*}{\eta}_{B13}c\omega_A t-\overset{*}{\eta}_{B7}\overset{*}{\eta}_{B9}s\omega_A t$$
$$+\overset{*}{\eta}_{B7}\overset{*}{\eta}_{B13}c\omega_A t-\overset{*}{\eta}_{B1}\overset{*}{\eta}_{B11}s\omega_A t-\overset{*}{\eta}_{B5}\overset{*}{\eta}_{B11}s\omega_A t+\overset{*}{\eta}_{B1}\overset{*}{\eta}_{B15}c\omega_A t+\overset{*}{\eta}_{B5}\overset{*}{\eta}_{B15}c\omega_A t]$$

$$J_{B32} = 2\bar{\phi}_{zB1}(\overset{*}{\eta}_{B11}s\omega_A t-\overset{*}{\eta}_{B15}c\omega_A t)+2\bar{\phi}_{B11}[-\overset{*}{\eta}_{B3}\overset{*}{\eta}_{B9}c\omega_A t-\overset{*}{\eta}_{B3}\overset{*}{\eta}_{B13}s\omega_A t+\overset{*}{\eta}_{B7}\overset{*}{\eta}_{B9}c\omega_A t$$
$$+\overset{*}{\eta}_{B7}\overset{*}{\eta}_{B13}s\omega_A t-\overset{*}{\eta}_{B1}\overset{*}{\eta}_{B11}c\omega_A t+\overset{*}{\eta}_{B5}\overset{*}{\eta}_{B11}c\omega_A t-\overset{*}{\eta}_{B1}\overset{*}{\eta}_{B15}s\omega_A t+\overset{*}{\eta}_{B5}\overset{*}{\eta}_{B15}s\omega_A t]$$

$$J_{B33} = 4B_B+2\bar{\phi}_{B11}(2\overset{*}{\eta}{}_{B1}^{2}+2\overset{*}{\eta}{}_{B5}^{2}+\overset{*}{\eta}{}_{B9}^{2}+\overset{*}{\eta}{}_{B13}^{2})$$

(B6)

$$M_B = 2\bar{\phi}_{B11}[2\dot{\eta}^{*2}_{B1}+2\dot{\eta}^{*2}_{B3}+2\dot{\eta}^{*2}_{B5}+2\dot{\eta}^{*2}_{B7}+\dot{\eta}^{*2}_{B9}+\dot{\eta}^{*2}_{B11}+\dot{\eta}^{*2}_{B13}+\dot{\eta}^{*2}_{B15}+\omega_A^2(\eta^{*2}_{B9}+\eta^{*2}_{B11}+\eta^{*2}_{B13}+\eta^{*2}_{B15})$$

$$+2\omega_A(\dot{\eta}^*_{B9}\eta^*_{B13}-\dot{\eta}^*_{B13}\eta^*_{B9}+\dot{\eta}^*_{B11}\eta^*_{B15}-\dot{\eta}^*_{B15}\eta^*_{B11})] \qquad \text{(B7)}$$

$$\dot{\bar{u}}_B = \begin{Bmatrix} 2\bar{\phi}_{B1}[\dot{\eta}^*_{B9}s\omega_At-\dot{\eta}^*_{B13}c\omega_At+\omega_A(\eta^*_{B9}c\omega_At+\eta^*_{B13}s\omega_At)] \\ 2\bar{\phi}_{B1}[\dot{\eta}^*_{B9}c\omega_At+\dot{\eta}^*_{B13}s\omega_At+\omega_A(-\eta^*_{B9}s\omega_At+\eta^*_{B13}c\omega_At)] \\ 4\bar{\phi}_{B1}\dot{\eta}_{B3} \end{Bmatrix} \qquad \text{(B8)}$$

$$\bar{\tilde{r}}_B = \begin{bmatrix} 0 & -4\bar{\phi}_{B1}\eta^*_{B3} & 2\bar{\phi}_{B1}(\eta^*_{B9}c\omega_At+\eta^*_{B13}s\omega_At) \\ 4\bar{\phi}_{B1}\eta^*_{B3} & 0 & 2\bar{\phi}_{B1}(-\eta^*_{B9}s\omega_At+\eta^*_{B13}c\omega_At) \\ -2\bar{\phi}_{B1}(\eta^*_{B9}c\omega_At+\eta^*_{B13}s\omega_At) & 2\bar{\phi}_{B1}(\eta^*_{B9}s\omega_At-\eta^*_{B13}c\omega_At) & 0 \end{bmatrix} \qquad \text{(B9)}$$

The components of the vector $\bar{H}_B$ are:

$$\begin{aligned}\bar{H}_{B1} = {} & 2\bar{\phi}_{zB1}[s\omega_At(\dot{\eta}^*_{B11}+\omega_A\eta^*_{B15})-c\omega_At(\dot{\eta}^*_{B15}-\omega_A\eta^*_{B11})]+2\bar{\phi}_{B11}\{c\omega_At[\dot{\eta}^*_{B1}\eta^*_{B11} \\ & -\dot{\eta}^*_{B11}\eta^*_{B1}-\dot{\eta}^*_{B3}\eta^*_{B9}+\dot{\eta}^*_{B9}\eta^*_{B3}-\dot{\eta}^*_{B5}\eta^*_{B11}+\dot{\eta}^*_{B11}\eta^*_{B5}+\dot{\eta}^*_{B7}\eta^*_{B9}-\dot{\eta}^*_{B9}\eta^*_{B7} \\ & +\omega_A(\eta^*_{B3}\eta^*_{B13}-\eta^*_{B1}\eta^*_{B15}-\eta^*_{B7}\eta^*_{B13}+\eta^*_{B5}\eta^*_{B15})]+s\omega_At[\dot{\eta}^*_{B1}\eta^*_{B15}-\dot{\eta}^*_{B15}\eta^*_{B1} \\ & -\dot{\eta}^*_{B3}\eta^*_{B13}+\dot{\eta}^*_{B13}\eta^*_{B3}-\dot{\eta}^*_{B5}\eta^*_{B15}+\dot{\eta}^*_{B15}\eta^*_{B5}+\dot{\eta}^*_{B7}\eta^*_{B13}-\dot{\eta}^*_{B13}\eta^*_{B7}+\omega_A(\eta^*_{B1}\eta^*_{B11} \\ & -\eta^*_{B3}\eta^*_{B9}+\eta^*_{B7}\eta^*_{B9}-\eta^*_{B5}\eta^*_{B11})]\}\end{aligned}$$

$$\begin{aligned}\bar{H}_{B2} = {} & 2\bar{\phi}_{zB1}[s\omega_At(\dot{\eta}^*_{B15}-\omega_A\eta^*_{B11})+c\omega_At(\dot{\eta}^*_{B11}+\omega_A\eta^*_{B15})]+2\bar{\phi}_{B11}\{c\omega_At[\dot{\eta}^*_{B1}\eta^*_{B15} \\ & -\dot{\eta}^*_{B15}\eta^*_{B1}-\dot{\eta}^*_{B3}\eta^*_{B13}+\dot{\eta}^*_{B13}\eta^*_{B3}+\dot{\eta}^*_{B5}\eta^*_{B15}-\dot{\eta}^*_{B15}\eta^*_{B5}-\dot{\eta}^*_{B7}\eta^*_{B13}+\dot{\eta}^*_{B13}\eta^*_{B7} \\ & +\omega_A(\eta^*_{B1}\eta^*_{B11}-\eta^*_{B3}\eta^*_{B9}-\eta^*_{B7}\eta^*_{B9}+\eta^*_{B5}\eta^*_{B11})]+s\omega_At[-\dot{\eta}^*_{B1}\eta^*_{B11}+\dot{\eta}^*_{B11}\eta^*_{B1} \\ & -\dot{\eta}^*_{B9}\eta^*_{B3}+\dot{\eta}^*_{B3}\eta^*_{B9}-\dot{\eta}^*_{B5}\eta^*_{B11}+\dot{\eta}^*_{B11}\eta^*_{B5}-\dot{\eta}^*_{B7}\eta^*_{B9}+\dot{\eta}^*_{B9}\eta^*_{B7}+\omega_A(-\eta^*_{B3}\eta^*_{B13} \\ & +\eta^*_{B1}\eta^*_{B15}-\eta^*_{B7}\eta^*_{B13}+\eta^*_{B5}\eta^*_{B15})]\}\end{aligned}$$

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$$\bar{\bar{H}}_{B3} = -4\bar{\phi}_{zB1}\dot{\eta}^*_{B1} \tag{B10}$$

where $\bar{\phi}_{B11}$, $\bar{\phi}_{zB1}$, $\bar{\phi}_{B1}$ and B_B were defined in Appendix A.

Appendix C

Throughout this appendix any element of a matrix which is not specified is equal to zero.

i. The matrix M

The elements M_{rs} (r,s = 1,2,...,19) of M have the following values:

$M_{11} = M_{22} = M_{33} = m_C + m_A + 4m_B$

$M_{15} = M_{51} = -M_{24} = -M_{42} = \frac{1}{2} m_A L_S + 4m_B L_S$

$M_{44} = B_A + 4m_B L_S^2 + 2B_B$

$M_{35} = M_{53} = -M_{26} = -M_{62} = m_A x_{CA} + 4m_B x_{CA}$

$M_{55} = B_C + m_A x_{CA}^2 + B_A + 4m_B(x_{CA}^2 + L_S^2) + 2B_B$

$M_{66} = B_C + m_A x_{CA}^2 + 4m_B x_{CA}^2 + 4B_B$

$M_{46} = M_{64} = (\frac{1}{2} m_A + 4m_B)x_{CA} L_S$

$M_{77} = \bar{\phi}_{C11} + (m_A + 4m_B)\phi_{C1A}^2 + 4\phi'^2_{C1A} B_B$

$M_{67} = M_{76} = \bar{\phi}_{xC1} -(m_A + 4m_B)x_{CA}\phi_{C1A} + 4\phi'_{C1A} B_B$

$M_{47} = M_{74} = -(\frac{1}{2} m_A + 4m_B)\phi_{C1A} L_S$

$M_{27} = M_{72} = M_{38} = M_{83} = \bar{\phi}_{C1} + (m_A + 4m_B)\phi_{C1A}$

$M_{88} = \bar{\phi}_{C11} + (m_A + 4m_B)\phi_{C1A}^2 + (B_A + 4m_B L_S^2 + 2B_B)\phi'^2_{C1A}$

$M_{58} = M_{85} = -\bar{\phi}_{xC1} + (m_A + 4m_B)x_{CA}\phi_{C1A} - (B_A + 4m_B L_S^2 + 2B_B)\phi'_{C1A}$

$M_{18} = M_{81} = - (\frac{1}{2} m_A + 4m_B)\phi'_{C1A} L_S$

$M_{99} = \bar{\phi}_{A11} + 4m_B\phi_{A1B}^2 + 2B_B\phi'^2_{A1B}$

$M_{89} = M_{98} = -(\bar{\phi}_{zA1} + 4m_B L_S\phi_{A1B} + 2B_B\phi'_{A1B})\phi'_{C1A}$

$M_{59} = M_{95} = - M_{4,10} = - M_{10,4} = \bar{\phi}_{zA1} + 4m_B L_S\phi_{A1B} + 2B_B\phi'_{A1B}$

$M_{91} = M_{19} = M_{2,10} = M_{10,2} = \bar{\phi}_{A1} + 4m_B\phi_{A1B}$

$M_{10,10} = \bar{\phi}_{A11} + 4m_B\phi_{A1B}^2 + 2\phi'^2_{A1B} B_B$

$M_{7,10} = M_{10,7} = (\bar{\phi}_{A1} + 4m_B\phi_{A1B})\phi_{C1A}$

$M_{6,10} = M_{10,6} = -(m_A\bar{\phi}_{A1} + 4m_B\phi_{A1B})x_{CA}$

$M_{11,11} = 4\Theta_{A1B}^2 B_B$

$M_{7,11} = M_{11,7} = 4\phi'_{C1A}\Theta_{A1B} B_B$

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$M_{6,11} = M_{11,6} = 4\Theta_{A1B}B_B$

$M_{12,12} = M_{13,13} = M_{14,14} = M_{15,15} = 4\bar{\phi}_{B11}$

$M_{7,12} = M_{12,7} = 4\phi'_{C1A}\bar{\phi}_{zB1}$

$M_{6,12} = M_{12,6} = 4\bar{\phi}_{zB1}$

$M_{8,13} = M_{13,8} = 4\phi_{C1A}\bar{\phi}_{B1}$

$M_{5,13} = M_{13,5} = 4x_{CA}\bar{\phi}_{B1}$

$M_{3,13} = M_{13,3} = 4\bar{\phi}_{B1}$

$M_{16,16} = M_{17,17} = M_{18,18} = M_{19,19} = 2\bar{\phi}_{B11}$

$M_{10,16} = M_{16,10} = -M_{9,18} = -M_{18,9} = 2\phi_{A1B}\bar{\phi}_{B1}$

$M_{7,16} = M_{16,7} = 2\phi_{C1A}\bar{\phi}_{B1}$

$M_{8,17} = M_{17,8} = 2\phi'_{C1A}\bar{\phi}_{zB1}$

$M_{9,17} = M_{17,9} = M_{10,19} = M_{19,10} = -\ 2\phi'_{A1B}\bar{\phi}_{zB1}$

$M_{8,18} = M_{18,8} = 2\phi'_{C1A}L_S\bar{\phi}_{B1}$

$M_{6,16} = M_{16,6} = -\ 2x_{CA}\bar{\phi}_{B1}$

$M_{4,16} = M_{16,4} = M_{5,18} = M_{18,5} = -\ 2L_S\bar{\phi}_{B1}$

$M_{2,16} = M_{16,2} = -\ M_{1,18} = -\ M_{18,1} = 2\bar{\phi}_{B1}$

$M_{5,17} = M_{17,5} = -\ M_{4,19} = -\ M_{19,4} = -\ 2\bar{\phi}_{zB1}$

where we have used the definitions

$$\phi_{C1A} = \phi_{C1}(x_C)\big|_{x_C = -x_{CA}} \qquad \phi'_{C1A} = \frac{d\phi_{C1}(x_C)}{\partial x_C}\bigg|_{x_C = -x_{CA}}$$

$$\phi_{A1B} = \phi_{A1}(z_A)\big|_{z_A = L_S} \qquad \phi'_{A1B} = \frac{d\phi_{A1}(z_A)}{dz_A}\bigg|_{z_A = L_S}$$

$$\Theta_{A1B} = \Theta_{A1}(z_A)\big|_{z_A = L_S}$$

All other quantities are defined in Appendix A or the text.

ii. The matrix F/ω_A

The elements F_{rs} $(r,s = 1,2,\ldots,19)$ of F/ω_A have the following values:

$F_{41} = F_{52} = (\frac{1}{2}\, m_A + 4m_B)L_S$

$F_{45} = B_A + 4m_B L_S^2 + 2B_B$

$F_{48} = -\,(B_A + 4m_B L_S^2 + 2B_B)\phi'_{C1A}$

$F_{49} = F_{5,10} = \bar{\phi}_{zA1} + 4m_B L_S \phi_{A1B} + 2\phi'_{A1B} B_B$

$F_{4,17} = F_{5,19} = -\,2\bar{\phi}_{zB1}$

$F_{4,18} = -\,F_{5,16} = -\,2\bar{\phi}_{B1} L_S$

$F_{54} = -\,B_A - 4m_B L_S^2 + 2B_B$

$F_{56} = -\,(\frac{1}{2}\, m_A + 4m_B)x_{CA} L_S$

$F_{57} = (\frac{1}{2}\, m_A + 4m_B)\phi_{C1A} L_S$

$F_{82} = -\,(\frac{1}{2}\, m_A + 4m_B)\phi'_{C1A} L_S$

$F_{84} = (B_A + 4m_B L_S^2 - 2B_B)\phi'_{C1A}$

$F_{86} = (\frac{1}{2}\, m_A + 4m_B)x_{CA}\phi'_{C1A} L_S$

$F_{87} = -\,(\frac{1}{2}\, m_A + 4m_B)\phi_{C1A}\phi'_{C1A} L_S$

$F_{8,10} = -\,(\bar{\phi}_{zA1} + 4m_B L_S \phi_{A1B} + 2\phi'_{A1B} B_B)\phi'_{C1A}$

$F_{8,16} = -\,2\phi'_{C1A} L_S \bar{\phi}_{B1}$

$F_{8,19} = 2\phi'_{C1A}\bar{\phi}_{zB1}$

$F_{92} = -\,F_{10,1} = -\,4m_B \phi_{A1B}$

$F_{94} = F_{10,5} = 4m_B \phi_{A1B} L_S + 4\phi'_{A1B} B_B$

$F_{96} = 4m_B x_{CA}\phi_{A1B}$

$F_{97} = -\,4m_B \phi_{C1A}\phi_{A1B}$

$F_{9,10} = -\,F_{10,9} = -\,4m_B \phi_{A1B}^2$

$F_{9,16} = F_{10,18} = -\,2\phi_{A1B}\bar{\phi}_{B1}$

$F_{10,8} = -\,4(m_B L_S \phi_{A1B} + \phi'_{A1B} B_B)\phi'_{C1A}$

$F_{17,4} = F_{19,5} = -\,4\bar{\phi}_{zB1}$

$F_{16,18} = F_{17,19} = -\,F_{18,16} = -\,F_{19,17} = -\,2\bar{\phi}_{B11}$

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$$F_{17,10} = -F_{19,9} = 4\phi'_{A1B}\bar{\Phi}_{zB1}$$

$$F_{19,8} = 4\phi'_{C1A}\bar{\Phi}_{zB1}$$

iii. The matrix K_T/ω_A^2

The elements K_{Trs} $(r,s = 1,2,\ldots,19)$ of K_T/ω_A^2 have the following values:

$$K_{T44} = K_{T55} = B_A + 4m_BL_S^2 - 2B_B$$

$$K_{T4,10} = K_{T10,4} = -K_{T59} = -K_{T95} = 4m_B\phi_{A1B}L_S + 4B_B\phi'_{A1B}$$

$$K_{T4,19} = K_{T19,4} = -K_{T5,17} = -K_{T17,5} = -4\bar{\Phi}_{zB1}$$

$$K_{T58} = K_{T85} = -(B_A + 4m_BL_S^2 - 2B_B)\phi'_{C1A}$$

$$K_{T88} = (B_A + 4m_BL_S^2 + 2B_B)\phi'^2_{C1A}$$

$$K_{T89} = K_{T98} = 4m_B\phi'_{C1A}L_S\phi_{A1B} + 4\phi'_{C1A}\phi'_{A1B}B_B$$

$$K_{T8,17} = K_{T17,8} = -4\phi'_{C1A}\bar{\Phi}_{zB1}$$

$$K_{T99} = K_{T10,10} = 4m_B\phi^2_{A1B}$$

$$K_{T12,12} = K_{T14,14} = K_{T16,16} = K_{T18,18} = 4\bar{\Phi}_{B11}$$

$$K_{T17,17} = K_{T19,19} = 2\bar{\Phi}_{B11}$$

iv. The matrix K_V

The elements K_{Vrs} $(r,s = 1,2,\ldots,19)$ of K_V have the following values:

$$K_{V77} = K_{V88} = \int_{x_C} EI_C\left(\frac{d^2\phi_{C1}(x_C)}{dx_C^{\,2}}\right)^2 dx_C$$

$$K_{V99} = K_{V10,10} = \int_0^{L_S} EI_A\left(\frac{d^2\phi_{A1}(z_A)}{dz_A^{\,2}}\right)^2 dz_A$$

$$K_{V11,11} = \int_0^{L_S} GJ\left(\frac{d\Theta_{A1}(z_A)}{dz_A}\right)^2 dz_A$$

$$K_{V12,12} = K_{V13,13} = K_{V14,14} = K_{V15,15} = K_{V16,16} = 2K_{V17,17} = 2K_{V18,18} =$$

$$2K_{V19,19} = \int_0^{L_B} [EI_B\left(\frac{d^2\phi_{B1}(\zeta)}{d\zeta^2}\right)^2 + \omega_A^2\rho_B(L_B^2 - \zeta^2)\left(\frac{d\phi_{B1}(\zeta)}{d\zeta}\right)^2]d\zeta$$

Appendix D

The constraint equations are

$$(m_C + m_A + 4m_B)\dot{w}_{OCx} + (\frac{1}{2} m_A + 4m_B)L_S(\dot{\theta}_{C2} + \omega_A\theta_{C1}) - (\frac{1}{2} m_A + 4m_B)\phi'_{C1A}L_S\dot{\eta}_{C3}$$

$$+ (\bar{\phi}_{A1} + 4m_B\phi_{A1B})\dot{\eta}^*_{A1} - 2\bar{\phi}_{B1}\dot{\eta}^*_{B13} + 4m_B\phi_{A1B}\omega_A\eta^*_{AB} = X_1 \qquad \text{(D1)}$$

$$(m_C + m_A + 4m_B)w_{OCy} - (\frac{1}{2} m_A + 4m_B)L_S(\dot{\theta}_{C1} - \omega_A\theta_{C2}) - (m_A + 4m_B)x_{CA}\dot{\theta}_{C3}$$

$$+ (\bar{\phi}_{C1} + m_A\phi_{C1A} + 4m_B\phi_{C1A})\dot{\eta}_{C1} - (\frac{1}{2} m_A + 4m_B)L_S\phi'_{C1A}\omega_A\eta_{C3}$$

$$+ (\bar{\phi}_{A1} + 4m_B\phi_{A1B})\dot{\eta}^*_{A3} - 4m_B\phi_{A1B}\omega_A\eta^*_{A1} + 2\bar{\phi}_{B1}\dot{\eta}^*_{B9} = X_2 \qquad \text{(D2)}$$

$$(m_C + m_A + 4m_B)\dot{w}_{OCz} + (m_A + 4m_B)x_{CA}\dot{\theta}_{C2} + (\bar{\phi}_{C1} + m_A\phi_{C1A} + 4m_B\phi_{C1A})\dot{\eta}_{C3}$$

$$+ 4\bar{\phi}_{B1}\dot{\eta}^*_{B3} = X_3 \qquad \text{(D3)}$$

$$- (m_A + 4m_B)x_{CA}\dot{w}_{OCy} + (\frac{1}{2} m_A + 4m_B)x_{CA}L_S(\dot{\phi}_{C1} - \omega_A\theta_{C2}) + (B_C + m_A x_{CA}^2$$

$$+ 4m_B x_{CA}^2 + 4B_B)\dot{\theta}_{C3} + (\bar{\phi}_{xC1} - m_A x_{CA}\phi_{C1A} - 4m_B x_{CA}\phi_{C1A} + 4\phi'_{C1A}B_B)\dot{\eta}_{C1}$$

$$+ (\frac{1}{2} m_A + 4m_B)x_{CA}\phi'_{C1A}L_S\omega_A\eta_{C3} - (m_A\bar{\phi}_{A1} + 4m_B\phi_{A1B})x_{CA}\dot{\eta}^*_{A3}$$

$$+ 4m_B x_{CA}\phi_{A1B}\omega_A\eta^*_{A1} + 4\ \Theta_{A1B}B_B\dot{\eta}_{\phi 1} + 4\bar{\phi}_{zB1}\dot{\eta}^*_{B1} - x_{CA}\bar{\phi}_{B1}\dot{\eta}^*_{B9} = X_4 \qquad \text{((D4)}$$

where all quantities are defined in Appendix A, Appendix C and the text. The constants X_i $(i = 1,2,3,4)$ are determined by means of the initial conditions. Quite often they are taken as zero.